Imperial College
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Digital Control Systems

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Modules ELEC70090

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Built upon a 2009 version by A. Astolfi and D. Casagrande

Outline

- Introduction to digital control systems
- Z-transform: definition, properties and theorems
- Sampling and reconstruction
- The pulse transfer function
- Stability and performance
- Control design (discretization, W-plane, root locus and analytical methods)
- State space approach
- Optimal control (dynamic programming and LQR)
- Some advanced topics

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Reference books:

K. Ogata, Discrete-time control systems, Prentice-Hall

C.L. Phillips and H.T. Nagle, Digital control systems analysis and design, Prentice-Hall

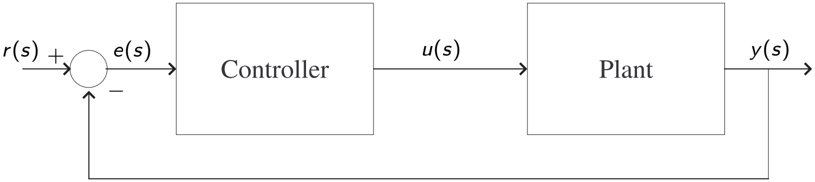
Any book of standard frequency-domain methods (Bode, Nyquist, ...)

Control systems are nowadays implemented by means of computers.

The use of digital controllers has several advantages and some disadvantages

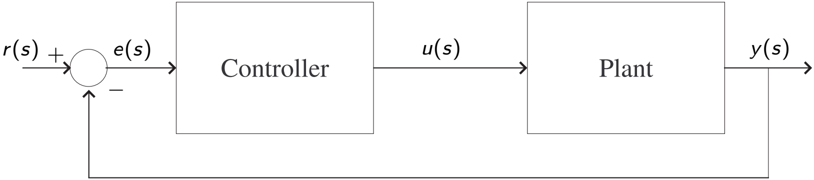
+ Controllers are easily implementable, and can be tuned on-line
+ Controllers are small and cheap
+ Complex controllers can be easily implemented
+ Controller can be used to implement monitoring and safety task
- The closed-loop system contains continuous-time components, discrete-time components and interfacing devices
- The analysis of the closed-loop system is often based on approximations
- Digital controllers are very sensitive to numerical errors
- Controller design is more involved and non-intuitive
- The notion of frequency for discrete-time systems is non-intuitive

Introduction – Computer controlled systems



The classical linear control loop (Plant: $G(s)$, Controller: $C(s)$).

Introduction – Computer controlled systems



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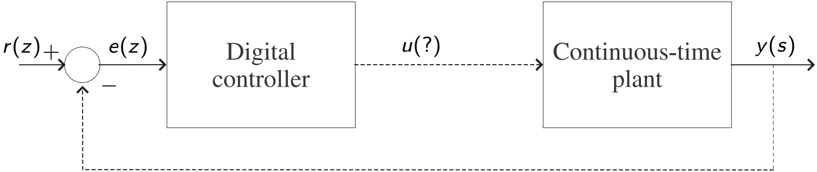
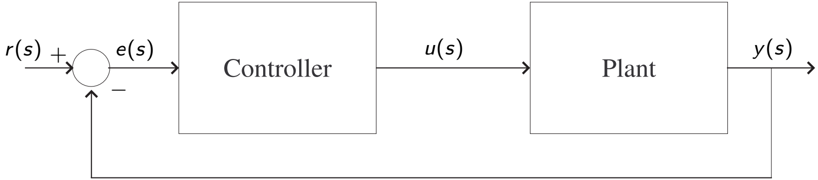


A basic digital control loop (Plant: $G(s)$, Controller: $C(z)$).

Introduction – Computer controlled systems



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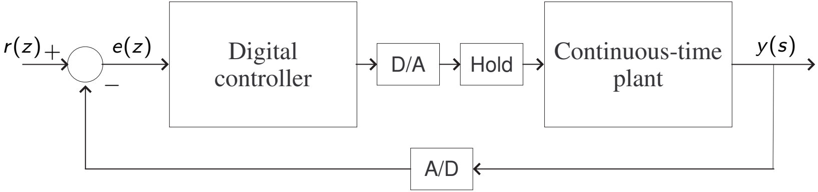
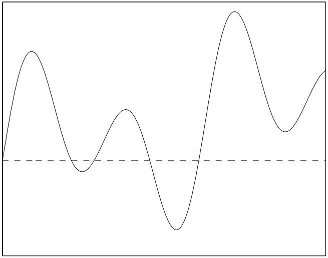
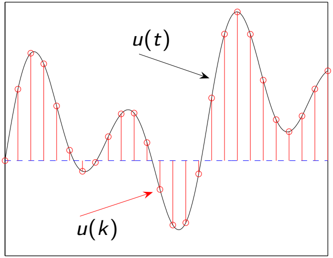


A digital control loop (Plant: $G(s)$, Controller: $C(z)$).

Introduction – A/D converter



Introduction – A/D converter

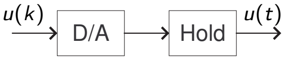


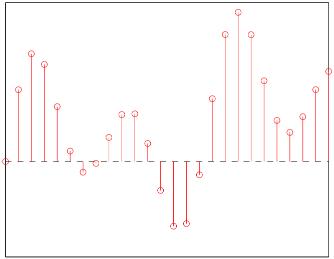
The A/D converter *transforms* a function of time $u(t)$ into a sequence $\{u(k)\}$. If the conversion is executed every T time instants then (with abuse of notation)

$$u(k) = u(kT),$$

for $k \in \mathbb{R}$ ($\mathbb{R}$ is the set of natural numbers, including zero). The time T is the sampling time. If the conversion is executed at times t_i, with $i \in \mathbb{R}$, then $u(k) = u(t_k)$.

Introduction – D/A converter with Hold





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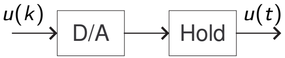
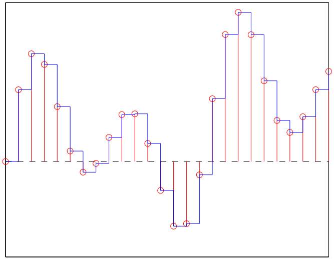
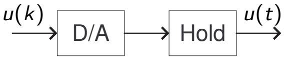
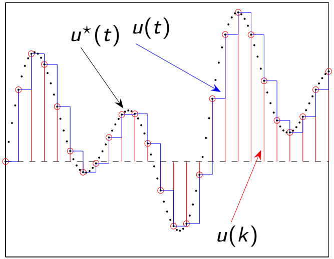




Introduction – D/A converter with Hold





Other hold devices, i.e. with different profiles of the output, can be used.

A/D and D/A conversion with Hold

The application of an A/D conversion, followed by a D/A conversion with Hold, to a signal $u(t)$ does not return the signal $u(t)$.

The A/D conversion associates the same sequence $u(k)$ to infinitely many signals $u(t)$.

The D/A and A/D conversions introduce other *distorsions*, such as quantization and delays, which are not discuss in-depth in this course.

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The z-transform

The z-transform is one of the mathematical tools used in the study of discrete-time systems. It plays a similar role to that of the Laplace transform for continuous-time systems.

A *discrete-time* (scalar) signal is a sequence of values

$$x(0), x(1), x(2), \cdots, x(k), \cdots$$

with $x(k) \in \mathbb{R}$. To denote the whole sequence we use the notation $\{x(k)\}$, where $k \in \mathbb{N}$.

A discrete-time signal may arise as the result of a sampling operation on a continuous-time signal, or as the result of an iterative process carried out, for example, by a computer.

The z-transform – Definition

Consider a sequence $\{x(k)\}$. The (one-sided) z-transform of the sequence, denoted $X(z)$, is defined as

$$X(z) = Z(\{x(k)\}) = Z(x(k)) = \sum_{k=0}^{\infty} x(k) z^{-k},$$

with $z \in \mathbb{C}$, whenever the indicated series exists.

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It is possible to define a two-sided z-transform for sequences $\{x(k)\}$, with $k \in \mathbb{N}$ ($\mathbb{N}$ is the set of integer numbers).

The one-sided z-transform coincides with the two-sided one for sequences $\{x(k)\}$ such that $x(k) = 0$, for all negative $k \in \mathbb{N}$.

In most engineering applications (and typically in control) it is sufficient to consider the one-sided z-transform and, often, the series defining the z-transform has a closed-form in the region of the complex plane in which the series converges.

The z-transform – Definition

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with $z \in \mathbb{C}$, whenever the indicated series exists.

The z-transform is a series in z^{-1}. Therefore, whenever the series converges, it converges outside the circle

$$|z| = R,$$

for some $R > 0$. The set $|z| > R$ is the *region of convergence* of the series, and R is the *radius of convergence*.

In practice it is not always necessary to specify the region of convergence of a certain z-transform, provided it is known that the series converges in some region.

The z-transform – Examples

Unit step function

$$x(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad \Rightarrow \quad \text{Sample time } T \quad \Rightarrow \quad x(k) = \begin{cases} 1 & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

$$\Downarrow$$

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1} \quad \Leftarrow \quad |z| > 1 \quad \Leftarrow \quad X(z) = 1 + z^{-1} + z^{-2} + \cdots$$

The z-transform – Examples

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Unit ramp function

$$x(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad \Rightarrow \quad \text{Sample time } T \quad \Rightarrow \quad x(k) = \begin{cases} kT & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

$$\Downarrow$$

$$X(z) = \frac{Tz^{-1}}{(1 - z^{-1})^2} = \frac{Tz}{(z-1)^2} \quad \Leftarrow \quad |z| > 1 \quad \Leftarrow \quad X(z) = T(z^{-1} + 2z^{-2} + \cdots)$$

The z-transform – Examples

Polynomial function

$$x(k) = a^k \quad \Rightarrow \quad X(z) = 1 + az^{-1} + a^2 z^{-2} + \cdots = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \qquad |z| > a$$

Exponential function

$$x(k) = e^{-akT} \quad \Rightarrow \quad X(z) = 1 + e^{-aT} z^{-1} + \cdots = \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}} \qquad |z| > e^{-aT}$$

Sinusoidal function

$$x(k) = \sin k\omega T = \frac{e^{jk\omega T} - e^{-jk\omega T}}{2j} \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad X(z) = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} \qquad |z| > 1$$

The z-transform – Properties (1/4)

Linearity. Let $X_1(z) = Z(x_1(k))$, $X_2(z) = Z(x_2(k))$, $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$. Then

$$Z(\alpha_1 x_1(k) + \alpha_2 x_2(k)) = \alpha_1 X_1(z) + \alpha_2 X_2(z).$$

Multiplication by a^k. Let $X(z) = Z(x(k))$ and $a \in \mathbb{C}$. Then

$$Z(a^k x(k)) = X\left(\frac{z}{a}\right).$$

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Proof. Note that

$$Z(a^k x(k)) = \sum_{k=0}^{\infty} a^k x(k) z^{-k} = \sum_{k=0}^{\infty} x(k) \left(\frac{z}{a}\right)^{-k} = X\left(\frac{z}{a}\right).$$

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Multiplication by a^k. Let $X(z) = Z(x(k))$ and $a \in \mathbb{C}$. Then

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Shifting Theorem. Let $X(z) = Z(x(k))$, $n \in \mathbb{N}$ and $x(k) = 0$, for $k < 0$. Then

$$Z(x(k-n)) = z^{-n} X(z).$$

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Proof. Note that

$$Z(x(k-n)) = \sum_{k=0}^{\infty} x(k-n) z^{-k} = z^{-n} \sum_{k=0}^{\infty} x(k-n) z^{-(k-n)} = z^{-n} \sum_{m=0}^{\infty} x(m) z^{-m} = z^{-n} X(z).$$

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$$Z(x(k-n)) = z^{-n} X(z).$$

In addition

$$Z(x(k+n)) = z^n \left[X(z) - \sum_{k=0}^{n-1} x(k) z^{-k} \right].$$

Note that $x(k+n)$ is the sequence shifted to the left (with a forward time shift), and $x(k-n)$ is the sequence shifted to the right (with a backward time shift).

The z-transform – Properties (2/4)

Backward difference. The (first) backward difference between $x(k)$ and $x(k-1)$ is defined as

$$\nabla x(k) = x(k) - x(k-1).$$

Then

$$Z(\nabla x(k)) = Z(x(k)) - Z(x(k-1)) = X(z) - z^{-1}X(z) = (1 - z^{-1})X(z).$$

Forward difference. The (first) forward difference between $x(k+1)$ and $x(k)$ is defined as

$$\Delta x(k) = x(k+1) - x(k).$$

Then

$$Z(\Delta x(k)) = Z(x(k+1)) - Z(x(k)) = (zX(z) - zx(0)) - X(z) = (z-1)X(z) - zx(0).$$

The z-transform – Properties (3/4)

Complex translation Theorem. Let $X(z) = Z(x(k))$ and $\alpha \in \mathbb{C}$. Then

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Proof. Note that

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Initial value Theorem. Let $X(z) = Z(x(k))$ and suppose that

$$\lim_{z\to\infty} X(z)$$

exists. Then

$$x(0) = \lim_{z\to\infty} X(z).$$

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Proof. Note that

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} = x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots ,$$

hence, letting $z \to \infty$ yields the claim (since the limit exists).

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Final value Theorem. Let $X(z) = Z(x(k))$ and suppose that all poles of $X(z)$ are in D^- (D^- denotes the interior of the unity circle), with the possible exception of a single pole at $z = 1$. Then

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}(1 - z^{-1})X(z).$$

The z-transform – Properties (4/4)

Complex differentiation. Let $X(z) = Z(x(k))$. Then

$$Z(kx(k)) = -z\frac{d}{dz}X(z),$$

and the derivative $\frac{d}{dz}X(z)$ converges in the same region as $X(z)$.

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Proof. Note that

$$\frac{d}{dz}X(z) = \sum_{k=0}^{\infty}(-k)x(k)z^{-k-1}$$

hence

$$-z\frac{d}{dz}X(z) = \sum_{k=0}^{\infty}kx(k)z^{-k} = Z(kx(k)).$$

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Complex integration. Let $X(z) = Z(x(k))$ and $g(k) = \frac{x(k)}{k}$. Assume $\lim_{k\to 0} g(k)$ is finite. Then

$$Z(g(k)) = \int_z^\infty \frac{X(\zeta)}{\zeta}d\zeta + \lim_{k\to 0} g(k)$$

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$$Z(g(k)) = \int_z^{\infty} \frac{X(\zeta)}{\zeta} d\zeta + \lim_{k\to 0} g(k)$$

Real convolution Theorem. Let $X_1(z) = Z(x_1(k))$ and $X_2(z) = Z(x_2(k))$. Then

$$X_1(z)X_2(z) = Z\left(\sum_{h=0}^{k} x_1(h)x_2(k-h)\right) = Z\left(\sum_{h=0}^{k} x_1(k-h)x_2(h)\right).$$

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The z-transform – The inverse transform

The z-transform is a *mapping* from a sequence $\{x(k)\}$ to a complex function $X(z)$.

This mapping is useful only if it is *invertible*, i.e. from a given $X(z)$ it is possible to find, in a unique way, the sequence $\{x(k)\}$ such that $Z(x(k)) = X(z)$.

The process of inversion generates a sequence at the sampling instants. **Note that the inverse z-transform of $X(z)$ yields a unique $x(k)$, but not a unique $x(t)$! No information on $x(t)$ outside the sampling times can be obtained.**

The sequence $\{x(k)\}$ is referred to as the inverse z-transform of $X(z)$, and we use the notation

$$\{x(k)\} = Z^{-1}(X(z)) \quad \text{or} \quad x(k) = Z^{-1}(X(z)).$$

The inverse z-transform of a complex function $X(z)$ can be computed by means of tables or of the following methods.

- The direct division method
- The partial fraction expansion method
- The computational method
- The inversion integral method

The z-transform – The direct division method

The inverse z-transform of the function $X(z)$ is obtained expanding $X(z)$ into a series in z^{-1}.

This method does not provide a closed-form expression for the sequence $\{x(k)\}$: it is useful to compute the first few elements of $\{x(k)\}$, and to *infer* some structure for the sequence.

The method is motivated by the definition of z-transform. In fact if

$$X(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_k}{z^k} + \cdots$$

then

$$x(0) = a_0 \qquad x(1) = a_1 \qquad x(2) = a_2 \quad \cdots \quad x(k) = a_k \quad \cdots$$

Example. Let

$$X(z) = \frac{10z+5}{(z-1)(z-1/5)} = \frac{10z^{-1}+5z^{-2}}{1-6/5z^{-1}+1/5z^{-2}} = 10z^{-1} + 17z^{-2} + 18.4z^{-3} + \cdots$$

hence

$$x(0) = 0 \qquad x(1) = 10 \qquad x(2) = 17 \qquad x(3) = 18.4 \quad \cdots$$

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then

$$x(0) = a_0 \qquad x(1) = a_1 \qquad x(2) = a_2 \quad \cdots \quad x(k) = a_k \quad \cdots$$

Example. Let

$$X(z) = \frac{Tz}{(z-1)^2} = \frac{Tz^{-1}}{1 - 2z^{-1} + z^{-2}} = Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \cdots$$

hence, for all $k \geq 0$,

$$x(k) = kT.$$

The z-transform – The computational method

The computational method allows to find the elements of the sequence $\{x(k)\}$ by means of an iterative process, which can be easily implemented in a computer.

Let, for example,

$$X(z) = \frac{a_1 z + a_0}{z^2 + b_1 z + b_0}$$

and note that

$$X(z) = \frac{a_1 z + a_0}{z^2 + b_1 z + b_0} U(z),$$

provided $U(z) = 1$, which implies

$$u(0) = 1 \qquad u(1) = 0 \qquad u(2) = 0 \qquad u(3) = 0 \quad \cdots$$

Recalling the shifting Theorem, we obtain

$$x(k+2) + b_1 x(k+1) + b_0 x(k) = a_1 u(k+1) + a_0 u(k),$$

which allows to compute, iteratively, $x(k)$, for $k \geq 2$, provided $x(1)$ and $x(0)$ are known.

The z-transform – The computational method

Recalling the shifting Theorem, we obtain

$$x(k+2) + b_1 x(k+1) + b_0 x(k) = a_1 u(k+1) + a_0 u(k),$$

which allows to compute, iteratively, $x(k)$, for $k \geq 2$, provided $x(1)$ and $x(0)$ are known.

Computation of $x(0)$

$$k = -2 \quad \Rightarrow \quad x(0) + b_1 x(-1) + b_2 x(-2) = a_1 u(-1) + a_0 u(-2)$$

$$x(-1) = x(-2) = 0 \qquad\qquad u(-1) = u(-2) = 0$$

$$\Downarrow$$

$$x(0) = 0$$

The z-transform – The computational method

Recalling the shifting Theorem, we obtain

$$x(k+2) + b_1 x(k+1) + b_0 x(k) = a_1 u(k+1) + a_0 u(k),$$

which allows to compute, iteratively, $x(k)$, for $k \geq 2$, provided $x(1)$ and $x(0)$ are known.

Computation of $x(1)$

$$k = -1 \quad \Rightarrow \quad x(1) + b_1 x(0) + b_2 x(-1) = a_1 u(0) + a_0 u(-1)$$

$$x(0) = x(-1) = 0 \qquad\qquad u(0) = 1 \quad u(-1) = 0$$

$$\Downarrow$$

$$x(1) = a_1$$

The z-transform – The computational method

Recalling the shifting Theorem, we obtain

$$x(k+2) + b_1 x(k+1) + b_0 x(k) = a_1 u(k+1) + a_0 u(k),$$

which allows to compute, iteratively, $x(k)$, for $k \geq 2$, provided $x(1)$ and $x(0)$ are known.

In summary

$$x(k+2) + b_1 x(k+1) + b_0 x(k) = a_1 u(k+1) + a_0 u(k)$$

with

$$x(0) = 0, \quad x(1) = a_1, \quad u(0) = 1, \quad u(k) = 0, k \geq 1.$$

Hence

$$x(2) = a_0 - b_1 a_1 \qquad x(3) = b_1(b_1 a_1 - a_0) - b_0 a_1 \qquad \cdots$$

The z-transform – The computational method

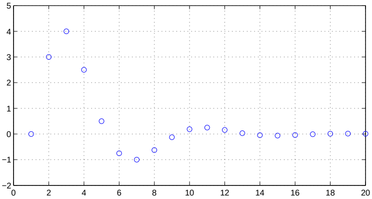
Recalling the shifting Theorem, we obtain

$$x(k+2) + b_1 x(k+1) + b_0 x(k) = a_1 u(k+1) + a_0 u(k),$$

which allows to compute, iteratively, $x(k)$, for $k \geq 2$, provided $x(1)$ and $x(0)$ are known.

Example. Let $X(z) = \dfrac{3z+1}{z^2 - z + 1/2}$

```
a0=1;a1=3;b0=1/2;b1=-1;
x0=0;x1=a1;u0=1;u1=0;
x=[x0,x1]; n=18;
for k = 1:1:n,
 x2=-b1*x1-b0*x0+a1*u1+a0*u0;
 x=[x,x2];
 x0=x1;x1=x2;u0=u1;
end
plot(x,'o');grid
```



Matlab code to compute and plot the first twenty values of the sequence $\{x(k)\} = Z^{-1}\left(\frac{3z+1}{z^2-z+1/2}\right)$.

The z-transform – The partial fraction expansion method

The partial fraction expansion method allows to obtain a closed-form expression for the sequence $\{x(k)\}$.

The method relies on the linearity of the z-transform and on the representation of the function $X(z)$ in a special form.

Let (assume $m \leq n$)

$$X(z) = \frac{n_0 z^m + n_1 z^{m-1} + \cdots + n_m}{z^n + d_1 z^{n-1} + \cdots + d_n} = \frac{n_0 z^m + n_1 z^{m-1} + \cdots + n_m}{(z - p_1)(z - p_2) \cdots (z - p_m)}.$$

Case simple poles: assume that $p_i \neq p_j$, for $i \neq j$, and that $p_i \neq 0$, for all i, and consider the function

$$\frac{X(z)}{z} = \frac{r_0}{z} + \frac{r_1}{z - p_1} + \frac{r_2}{z - p_2} + \cdots + \frac{r_n}{z - p_n},$$

where $r_0 = X(0)$ and

$$r_i = \lim_{z \to p_i} (z - p_i) \frac{X(z)}{z}.$$

The z-transform – The partial fraction expansion method

Then

$$\begin{aligned} X(z) &= r_0 + \frac{r_1 z}{z - p_1} + \frac{r_2 z}{z - p_2} + \cdots + \frac{r_n z}{z - p_n} \\ &= r_0 + \frac{r_1}{1 - p_1 z^{-1}} + \frac{r_2}{1 - p_2 z^{-1}} + \cdots + \frac{r_n}{1 - p_n z^{-1}} \end{aligned}$$

and recalling that

$$Z(\delta(k)) = 1 \qquad\qquad Z(a^k) = \frac{1}{1 - az^{-1}}$$

where $\delta(k) = 1$, for $k = 0$, and $\delta(k) = 0$, for $k \geq 1$, yields, for all $k \geq 0$,

$$x(k) = r_0\delta(k) + r_1 p_1^k + r_2 p_2^k + \cdots + r_n p_n^k.$$

Note that, since $X(z)$ has real coefficients, complex poles appear in conjugate pairs, hence the corresponding *residuals* are also conjugate pairs: the sequence $\{x(k)\}$ has real valued terms.
Case repeated poles: assume that $p_i = p$, for $i = 1, \ldots n$, and consider the function

$$\frac{X(z)}{z} = \frac{r_{i1}}{z - p} + \frac{r_{i2}}{(z - p)^1} + \cdots + \frac{r_{in}}{(z - p)^n},$$

where

$$r_{ij} = \frac{1}{(n - j)!} \lim_{z \to p} \frac{d^{n-j}}{dz^{n-j}} \left[(z - p)^n \frac{X(z)}{z} \right].$$

Example.

$$X(z) = \frac{10z+5}{(z-1)(z-1/5)}$$

$$\Downarrow$$

$$\frac{X(z)}{z} = \frac{10z+5}{z(z-1)(z-1/5)} = 25\frac{1}{z} + \frac{75}{4}\frac{1}{z-1} - \frac{175}{4}\frac{1}{z-1/5}$$

$$\Downarrow$$

$$X(z) = 25 + \frac{75}{4}\frac{z}{z-1} - \frac{175}{4}\frac{z}{z-1/5} = 25 + \frac{75}{4}\frac{1}{1-z^{-1}} - \frac{175}{4}\frac{1}{1-1/5z^{-1}}$$

$$\Downarrow$$

$$x(k) = 25\delta(k) + \frac{75}{4}1^k - \frac{175}{4}(1/5)^k \qquad k \geq 0$$

The z-transform – The partial fraction expansion method

Example.

$$X(z) = X(z) = \frac{3z+1}{z^2 - z + 1/2}$$

$$\Downarrow$$

$$\frac{X(z)}{z} = \frac{3z+1}{z(z-(1/2+1/2j))(z-(1/2-1/2j))} = \frac{2}{z} - \frac{1+4j}{z-(1/2+1/2j)} - \frac{1-4j}{z-(1/2-1/2j)}$$

$$\Downarrow$$

$$X(z) = 2 - \frac{(1+4j)z}{z-(1/2+1/2j)} - \frac{(1-4j)z}{z-(1/2-1/2j)} = 2 - \frac{1+4j}{1-(1/2+1/2j)z^{-1}} - \frac{1-4j}{1-(1/2-1/2j)z^{-1}}$$

$$\Downarrow$$

$$x(k) = 2\delta(k) - 2\left(\frac{1}{\sqrt{2}}\right)^k \cos\frac{k\pi}{4} + 8\left(\frac{1}{\sqrt{2}}\right)^k \sin\frac{k\pi}{4} \qquad k \geq 0$$

The z-transform – The inversion integral

The most general technique for finding inverse z-transforms relies upon the use of an *inversion integral*.

The theoretical justifications of this method are based on the theory of complex functions.

Let $X(z)$ be a z-transform and consider a circle $\mathcal{C}$ centered at the origin of the complex plane z and such that all poles of $X(z)z^{k-1}$ are inside $\mathcal{C}$.

Then

$$x(k) = \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{k-1} dz.$$

If the function $X(z)z^{k-1}$ has a finite number of poles, p_1 of order q_1, p_2 of order q_2, $\cdots$, p_n of order q_n, **with** $p_i \neq 0$ **for all** $i = 1, \ldots n$, then

$$\frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{k-1} dz = r_1 + r_2 + \cdots + r_n,$$

where

$$r_i = \frac{1}{(q_i - 1)!} \lim_{z \to p_i} \frac{d^{q_i - 1}}{dz^{q_i - 1}} \left[(z - p_i)^{q_i} X(z) z^{k-1} \right].$$

This formula is valid only if $X(z)z^{k-1}$ does not have poles at the origin!

Outline

- Introduction to digital control systems
- Z-transform: definition, properties and theorems
- Sampling and reconstruction
- The pulse transfer function
- Stability and performance
- Control design (discretization, W-plane, root locus and analytical methods)
- State space approach
- Optimal control (dynamic programming and LQR)
- Some advanced topics

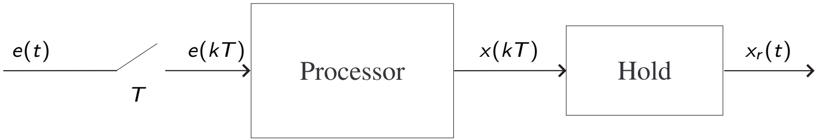
Sampling and reconstruction (1/5)

Discrete-time and continuous-time systems are interconnected by means of samplers (A/D converters) and holders (D/A converters).

The sampler converts a continuous-time signal into a sequence of samples taken at time $t = 0$, $t = T$, $t = 2T$, $\cdots$, where T is the sampling time.

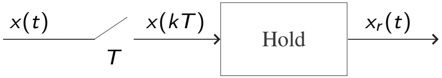
For $t \neq kT$ the sampler does not process information. Note that two signals may have equal samples but may be significantly different.

The main reason to transform the signal $x(t)$ into a sequence is that the latter can be easily processed by a computer. The processed sequence is then converted, by a hold device, into a continuous-time signal.



Sampler, processor, hold interconnection.

Sampling and reconstruction (1/5)



Consider the cascaded interconnection of a sampler and a hold device.

Suppose the hold device keeps its output at the value $x(kT)$ for all $t \in [kT, (k+1)T)$.
(This is the simplest possible hold, and it is named zero-order hold.)

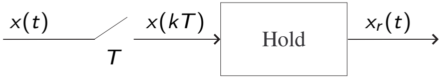
Consider the sequence $\{x(kT)\}$, and assume $x(kT) = 0$, for $k < 0$. Then

$$x_r(t) = \sum_{k=0}^{\infty} x(kT)\,[h(t-kT) - h(t-(k+1)T)]\,,$$

where

$$h(t-t_0) = \begin{cases} 0 & \text{if } t < t_0 \\ 1 & \text{if } t \geq t_0. \end{cases}$$

Sampling and reconstruction (1/5)



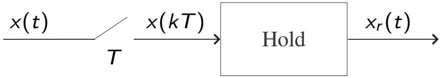
The signal $x_r(t)$ is Laplace transformable (recall that $\mathcal{L}(h(t-kT)) = \dfrac{e^{-kTs}}{s}$):

$$x_r(t) = \sum_{k=0}^{\infty} x(kT)\left[h(t-kT) - h(t-(k+1)T),\right]$$

$$\Downarrow$$

$$X_r(s) = \sum_{k=0}^{\infty} x(kT)\left[\frac{e^{-kTs} - e^{-(k+1)Ts}}{s}\right] = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

Sampling and reconstruction (1/5)

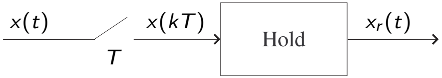


$X_r(s)$ can be expressed as the product of the functions

$$H_0(s) = \frac{1 - e^{-Ts}}{s} \qquad\qquad X^\star(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}.$$

The function $X^\star(s)$ is the Laplace transform of a signal $x^\star(t)$ which depends only upon the sequence of samples $\{x(kT)\}$.

Sampling and reconstruction (1/5)



The signal $X^\star(s)$ is inverse Laplace transformable:

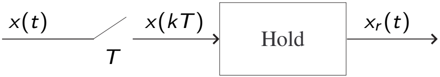
$$X^\star(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\Downarrow$$

$$x^\star(t) = \mathcal{L}^{-1}(X^\star(s)) = \sum_{k=0}^{\infty} x(kT)\delta(t - kT)$$

$\delta(t - kT)$ is a Dirac impulse of unity area centered at $t = kT$.

Sampling and reconstruction (1/5)



Let

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$

then

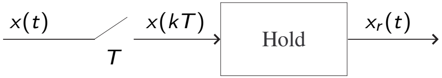
$$x^\star(t) = x(t)\delta_T(t),$$

i.e. $x^\star(t)$ is a sequence of Dirac impulses, modulated by the samples $x(kT)$.

The product of $x(t)$ with the signal $\delta_T(t)$ is called impulsive sampling of $x(t)$.

The impulsive sampler is an ideal model adequate for control analysis and design.

Sampling and reconstruction (1/5)

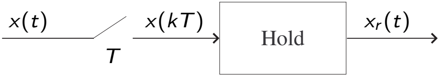


Applying the signal $X^\star(s)$ at the input of a system with transfer function $H_0(s)$ yields the signal $X_r(s)$, i.e.

$$X_r(s) = H_0(s)X^\star(s) = \frac{1-e^{-Ts}}{s}X^\star(s).$$

The transfer function $H_0(s)$ yields a correct mathematical description of a zero-order hold provided the sampler is replaced by an impulsive sampler.

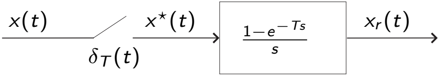
Sampling and reconstruction (1/5)



$$\Updownarrow$$

From an input-output perspective the cascaded interconnection of a sampler and a zero-order hold is equivalent to the cascaded interconnection of an impulsive sampler and the transfer function $H_0(s)$.

$$\Updownarrow$$



Sampling and reconstruction (2/5)

Consider again the signal

$$X^\star(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

and the transformation

$$z = e^{sT} \Leftrightarrow s = \frac{1}{T}\ln z.$$

Then

$$X^\star(s)\Big|_{s=\frac{1}{T}\ln z} = X^\star\left(\frac{1}{T}\ln z\right) = \sum_{k=0}^{\infty} x(kT)z^{-k} = X(z),$$

which provides a one-to-one relation between the Laplace transform of the impulsive signal $x^\star(t)$ and the z-transform of the sequence $\{x(kT)\}$.

Note that the z-transform $X(z)$ is (in general) a rational function, whereas $X^\star(s)$ is a trascendental function.

Sampling and reconstruction (3/5)

We now discuss the relation between $X^\star(s)$ and $X(s)$.

Since $x(t) = 0$ for all $t < 0$ we can write

$$x^\star(t) = x(t)\delta_T(t) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT) = x(t)\delta_T^e(t).$$

The signal $\delta_T^e(t)$ is periodic of period T, hence it can be represented with a Fourier series:

$$\delta_T^e(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_s t} \qquad \omega_s = \frac{2\pi}{T} \qquad c_n = \frac{1}{T}\int_0^T \delta_T^e(t)e^{-jn\omega_s t}dt = \frac{1}{T}$$

$$\Downarrow$$

$$x^\star(t) = x(t)\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t} = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)e^{jn\omega_s t}$$

Sampling and reconstruction (3/5)

$$x^\star(t) = x(t)\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t} = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)e^{jn\omega_s t}$$

$$\Downarrow$$

$$X^\star(s) = \mathcal{L}(x^\star(t)) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \mathcal{L}\left(x(t)e^{jn\omega_s t}\right) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(s - jn\omega_s)$$

The Laplace transform $X^\star(s)$ of the sampled signal is (disregarding the factor $\frac{1}{T}$)
the sum of infinitely many terms,
each obtained from the Laplace transform $X(s)$ of $x(t)$ translated by $jn\omega_s$.

Sampling and reconstruction (3/5)

The Laplace transform $X^\star(s)$ of the sampled signal is (disregarding the factor $\frac{1}{T}$) the sum of infinitely many terms, each obtained from the Laplace transform $X(s)$ of $x(t)$ translated by $jn\omega_s$.

$$\Downarrow$$

The Fourier transform $X^\star(j\omega)$ of the sampled signal is (disregarding the factor $\frac{1}{T}$) the sum of infinitely many terms, each obtained from the Fourier transform $X(j\omega)$ of $x(t)$ translated by $jn\omega_s$.

$$\Downarrow$$

$$X^\star(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega - jn\omega_s)$$

Shannon's Theorem. If $\omega_s = \dfrac{2\pi}{T} > 2\omega_x$, where ω_x is the highest frequency component of $x(t)$, then $x(t)$ can be reconstructed from the sampled signal $x^\star(t)$.

Sampling and reconstruction (3/5)

Shannon's Theorem. If $\omega_s = \dfrac{2\pi}{T} > 2\omega_x$, where ω_x is the highest frequency component of $x(t)$, then $x(t)$ can be reconstructed from the sampled signal $x^\star(t)$.

If Nyquist condition, i.e. $\omega_s > 2\omega_x$, holds then the reconstruction of the signal $x(t)$ is performed by a filter $G_I(j\omega)$ such that

$$G_I(j\omega) = \begin{cases} T & |\omega| \leq \frac{\omega_s}{2}, \\ 0 & \text{elsewhere.} \end{cases}$$

This filter is not physically realizable, since

$$g_I(t) = \mathcal{L}^{-1}(G_I(j\omega)) = \frac{\sin \omega_s t/2}{\omega_s t/2}$$

is such that $g_I(t) \neq 0$ for $t < 0$. In signal processing it may be possible to implement (approximations of) $G_I(j\omega)$, whereas this is impossible in real-time control applications.

This justifies the use, in control, of hold devices, which provide a very coarse approximation of $G_I(j\omega)$, but are causal and simple to implement.

Sampling and reconstruction (3/5)

Shannon's Theorem. If $\omega_s = \frac{2\pi}{T} > 2\omega_x$, where ω_x is the highest frequency component of $x(t)$, then $x(t)$ can be reconstructed from the sampled signal $x^\star(t)$.

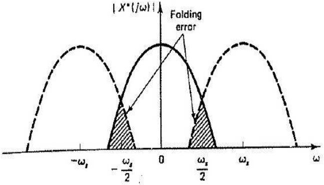
If Nyquist condition, i.e. $\omega_s > 2\omega_x$ does not hold it is not possible to reconstruct $x(t)$ (even with non-causal filters).

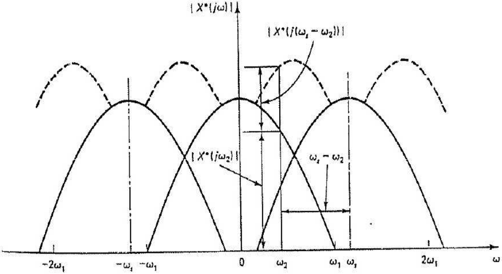
The presence of harmonic components of $x(t)$ with angular frequency larger than $\omega_s/2$ causes the phenomenon of *aliasing*.

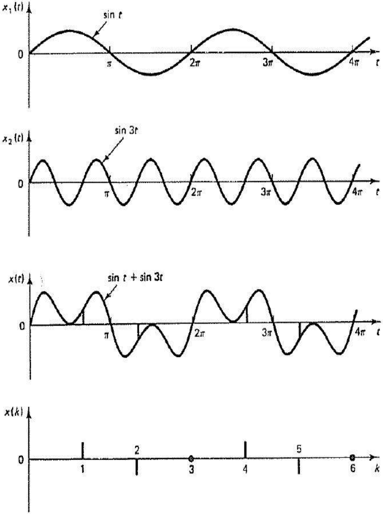
Note that

- signals encountered in signal processing and communications are often band limited, hence an adequate selection of the sampling frequency avoids aliasing;
- in control applications signals are not band-limited, hence it is essential to consider the effect of sampling on the system's performance.

Sampling and reconstruction (3/5)







Sampling and reconstruction (4/5)

Hold devices read the elements of the sequence $\{x(kT)\}$ and yield a continuous-time signal $x_r(t)$ which approximates, in some sense, the signal $x(t)$ which has generated the sequence.

Holds are obtained considering the Taylor series expansion of the signal $x(t)$ around $t = kT$:

$$x(t) = x(kT) + \left.\frac{dx(t)}{dt}\right|_{t=kT} (t - kT) + \cdots .$$

Note that, since only the elements of the sequence $\{x(kT)\}$ are available to the hold, the derivative of $x(t)$ is approximated as

$$\left.\frac{dx(t)}{dt}\right|_{t=kT} \approx \frac{x(kT) - x((k-1)T)}{T}.$$

The number of terms of the series exploited in the realization of the hold device determines the order of the hold. High-order holds are more precise, but more complex to implement.

Sampling and reconstruction (4/5)

Hold devices read the elements of the sequence $\{x(kT)\}$ and yield a continuous-time signal $x_r(t)$ which approximates, in some sense, the signal $x(t)$ which has generated the sequence.

Hold devices interface discrete-time systems and continuous-time systems, hence can be represented by a transfer function $H_r(s)$ only if the input sequence $\{x(kT)\}$ is interpreted as a sequence of modulated impulses.

This considerations yield the following results.

- Zero-order hold. As already discussed its transfer function and impulse response are

$$H_0(s) = \frac{1 - e^{-sT}}{s} \qquad h_0(t) = \begin{cases} 1 & \text{if } t \in [0, T), \\ 0 & \text{elsewhere.} \end{cases}$$

- First-order hold. Its transfer function and impulse response are

$$H_1(s) = \frac{1 + Ts}{T}\left(\frac{1 - e^{-sT}}{s}\right)^2 \qquad h_1(t) = \begin{cases} 1 + t/T & \text{if } t \in [0, T), \\ 1 - t/T & \text{if } t \in [T, 2T), \\ 0 & \text{elsewhere.} \end{cases}$$

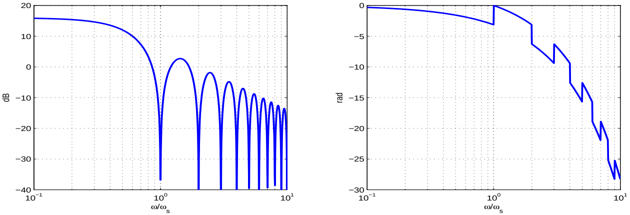
Sampling and reconstruction (4/5)

Zero-order hold

$$H_0(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = T\frac{\sin \omega T/2}{\omega T/2} e^{-j\omega T/2}$$

$$\Downarrow$$

$$|H_0(j\omega)| = T\left|\frac{\sin \omega T/2}{\omega T/2}\right| \qquad \angle H_0(j\omega) = \angle \sin \omega T/2 - \omega T/2$$



Frequency response of $H_0(s)$ ($T = 1$).

Sampling and reconstruction (4/5)

<u>Zero-order hold</u>

$$H_0(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = T\frac{\sin\omega T/2}{\omega T/2}e^{-j\omega T/2}$$

$$\Downarrow$$

$$|H_0(j\omega)| = T\left|\frac{\sin\omega T/2}{\omega T/2}\right| \qquad \angle H_0(j\omega) = \angle\sin\omega T/2 - \omega T/2$$

$H_0(0) = T$, i.e. the factor $1/T$ in the impulsive sampler is compensated by the hold.

For $\omega T/2 \ll 1$,

$$H_0(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = e^{-j\omega T/2}\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{j\omega} \approx Te^{-j\omega T/2}$$

i.e. at low frequency the hold is approximated by a delay of $T/2$ seconds.

$H_0(j\omega) = 0$ for $\omega T/2 = k\pi$, with $k \in \mathbb{R}$, i.e. for $\omega = k\omega_s$.

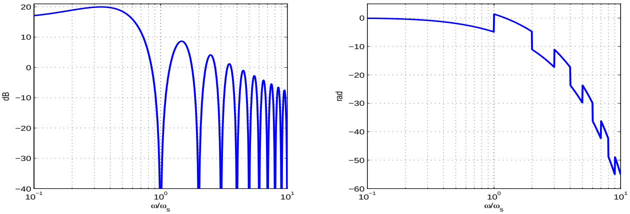
Sampling and reconstruction (4/5)

<u>First-order hold</u>

$$H_1(j\omega) = \left(\frac{1-e^{-j\omega T}}{j\omega}\right)^2 \frac{1+j\omega T}{T} = T\left(\frac{\sin\omega T/2}{\omega T/2}\right)^2 (1+j\omega T)e^{-j\omega T}$$

$$\Downarrow$$

$$|H_1(j\omega)| = T\left|\frac{\sin\omega T/2}{\omega T/2}\right|^2 \sqrt{1+\omega^2T^2} \qquad \angle H_1(j\omega) = 2\angle\sin\omega T/2 + \arctan\omega T - \omega T$$



Frequency response of $H_1(s)$ ($T=1$).

Sampling and reconstruction (4/5)

First-order hold

$$H_1(j\omega) = \left(\frac{1-e^{-j\omega T}}{j\omega}\right)^2 \frac{1+j\omega T}{T} = T\left(\frac{\sin\omega T/2}{\omega T/2}\right)^2 (1+j\omega T)e^{-j\omega T}$$

$$\Downarrow$$

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$H_1(0) = T$, i.e. the factor $1/T$ in the impulsive sampler is compensated by the hold.

For $\omega T/2 \ll 1$,

$$H_1(j\omega) \approx Te^{-j\omega T}$$

i.e. at low frequency the hold is approximated by a delay of T seconds.

$H_1(j\omega) = 0$ for $\omega T/2 = k\pi$, with $k \in \mathbb{R}$, i.e. for $\omega = k\omega_s$.

Sampling and reconstruction (5/5)

The Laplace transform $X^\star(s)$ of the sampled signal is related to the z-transform of the sequence $\{x(kT)\}$ of samples by the relation

$$X^\star(s) = X(z)\Big|_{z=e^{sT}}.$$

The equation

$$z = e^{sT}$$

establishes a relation between the two complex variables s and z, which allows to relate continuous-time properties to discrete-time properties.

Let $s = \sigma + j\omega$ and note that

$$z = e^{T(\sigma+j\omega)} = e^{T\sigma} e^{jT\omega} = e^{T\sigma} e^{jT\left(\omega+\frac{2k\pi}{T}\right)}.$$

Points on the s-plane with angular frequency which differs by an integer multiple of $\omega_s = \frac{2\pi}{T}$ are mapped into the same point in the z-plane.

Sampling and reconstruction (5/5)

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There are infinitely many values of s for each value of z.

Sampling and reconstruction (5/5)

$$z = e^{sT} \qquad\qquad s = \frac{1}{T}\log z$$

- Points on the s-plane with negative real part are mapped inside the unity circle of the z-plane: $Re(s) < 0 \Leftrightarrow |z| < 1$.
- Points on the s-plane with zero real part are mapped into the unity circle of the z-plane: $Re(s) = 0 \Leftrightarrow |z| = 1$.
- Points on the s-plane with positive real part are mapped outside the unity circle of the z-plane: $Re(s) > 0 \Leftrightarrow |z| > 1$.
- $\angle z = \omega T$, hence when ω varies from $-\omega_s/2$ to $\omega_s/2$ the phase of z varies from $-\pi$ to π.
- The phase of z varies from $-\pi$ to π for any change in ω from $-\omega_s/2 + k\omega_s$ to $\omega_s/2 + k\omega_s$, with $k \in \mathbb{R}$.

$$\Downarrow$$

It is possible to partition the s-plane into horizontal strips of width ω_s.

Sampling and reconstruction (5/5)

$$z = e^{sT} \qquad\qquad s = \frac{1}{T}\log z$$

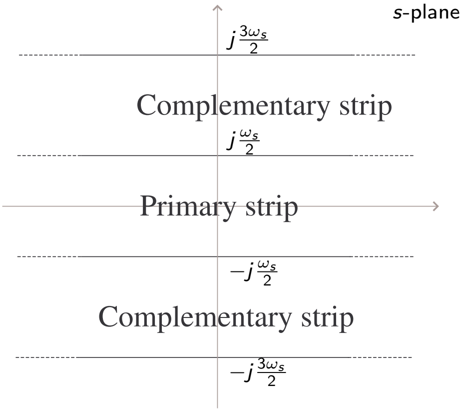
It is possible to partition the s-plane into horizontal strips of width ω_s.

- The strip between the horizontal lines $s = j\frac{\omega_s}{2}$ and $s = -j\frac{\omega_s}{2}$ is called *primary strip*.
- The strips between the horizontal lines $s = j\frac{\omega_s}{2} + k\omega_s$ and $s = -j\frac{\omega_s}{2} + k\omega s$, with $k \in \mathbb{R}$ and $k \neq 0$ are called *complementary strips*.

Sampling and reconstruction (5/5)

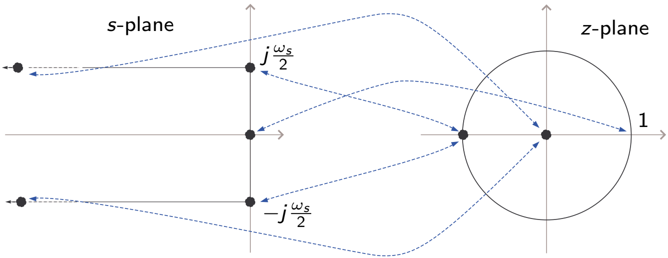
$$z = e^{sT} \qquad\qquad s = \frac{1}{T}\log z$$

It is possible to partition the s-plane into horizontal strips of width ω_s.



Sampling and reconstruction (5/5)

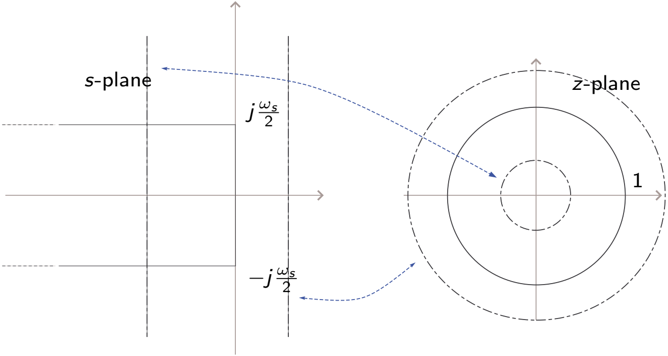
$$z = e^{sT} \qquad s = \frac{1}{T}\log z$$



Correspondence between the primary strip in the s-plane and the unity circle in the z-plane.

Sampling and reconstruction (5/5)

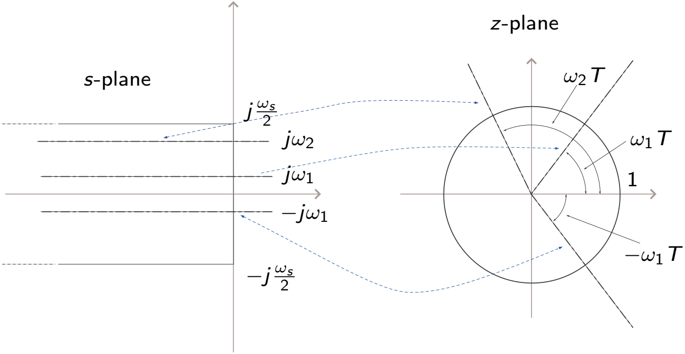
$$z = e^{sT} \qquad s = \frac{1}{T}\log z$$



Constant attenuation lines: $s = \sigma + j\omega$, with σ constant, $|z| = e^{\sigma T}$.

Sampling and reconstruction (5/5)

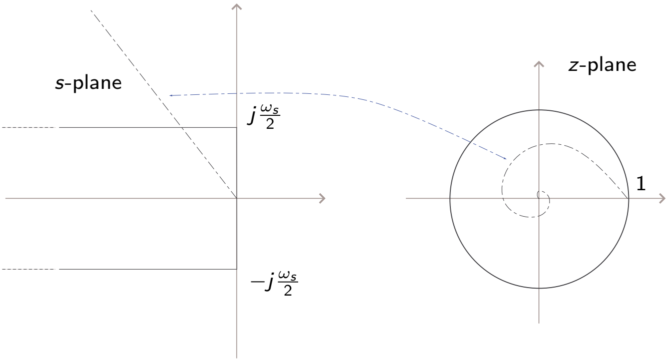
$$z = e^{sT} \qquad s = \frac{1}{T}\log z$$



Constant frequency lines: $s = \sigma + j\omega$, with ω constant, $|z| = e^{\sigma T} e^{j\omega T}$.

Sampling and reconstruction (5/5)

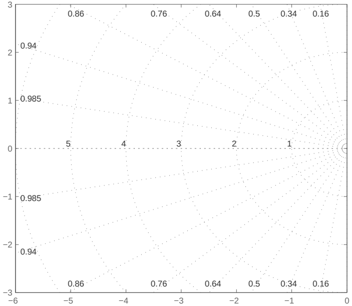
$$z = e^{sT} \qquad s = \frac{1}{T}\log z$$

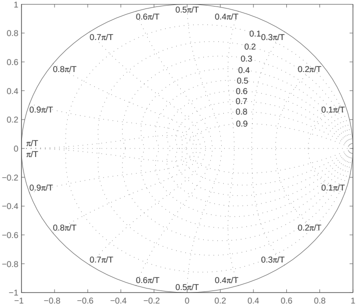


Constant damping-ratio lines: $s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$, with $\omega_n > 0$, $0 < \zeta < 1$ constant, $|z| = e^{-\zeta T\omega_n}$, $\angle z = \sqrt{1-\zeta^2}\omega_n T$. The locus on the z-plane is a logarithmic spiral.

Sampling and reconstruction (5/5)

$$z = e^{sT} \qquad\qquad s = \frac{1}{T}\log z$$





Constant ζ and ω_n loci in the s-plane (left) and in the z-plane (right).
Note that in the s-plane the loci are orthogonal.

Outline

- Introduction to digital control systems
- Z-transform: definition, properties and theorems
- Sampling and reconstruction
- The pulse transfer function
- Stability and performance
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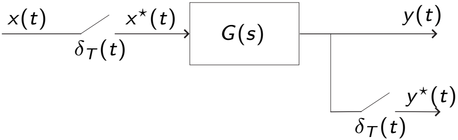
The pulse transfer function

Consider a continuous-time system *between* two impulsive samplers.



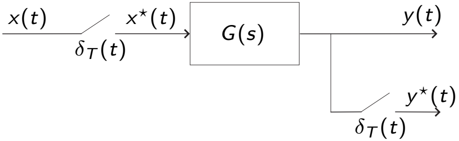
Let $x^\star(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$. Then (recall that $\mathcal{L}^{-1}(G(s)\mathcal{L}(\delta(t-kT))) = g(t-kT)$)

$$y(t) = \begin{cases} g(t)x(0) & \text{if } 0 \le t < T \\ g(t)x(0) + g(t-T)x(T) & \text{if } T \le t < 2T \\ \quad\vdots & \\ g(t)x(0) + g(t-T)x(T) + \cdots + g(t-kT)x(kT) & \text{if } kT \le t < (k+1)T \end{cases}$$

and since $g(t) = 0$ for $t < 0$

$$y(t) = g(t)x(0) + g(t-T)x(T) + \cdots + g(t-kT)x(kT) \qquad \text{if } 0 \le t < (k+1)T.$$

The pulse transfer function



$$x^\star(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT) \quad \Rightarrow \quad y(t) = \sum_{h=0}^{k} g(t-hT)x(hT) \quad \text{if } 0 \le t < (k+1)T$$

$$\Downarrow$$

$$y(kT) = \sum_{h=0}^{k} g(kT-hT)x(hT)$$

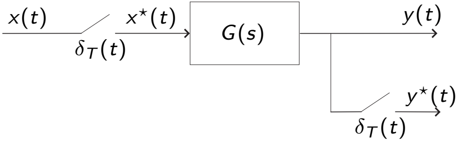
and, by the real convolution Theorem,

$$Y(z) = G(z)X(z),$$

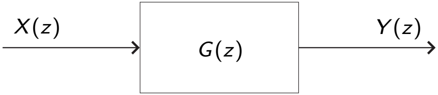
where

$$G(z) = Z(\mathcal{L}^{-1}(G(s))) \triangleq Z(G(s)).$$

The pulse transfer function



$\Downarrow$

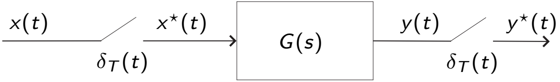


The discrete-time description provides information only at sampling instants.

The pulse transfer function - Block diagrams (1/4)

The definition of pulse transfer function is useful to obtain discrete-time transfer functions for interconnected systems in the presence of sampling devices.

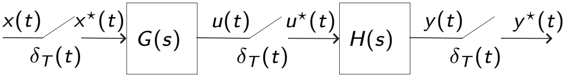
Note that the presence of samplers complicates the *algebra* of block diagrams, since the existence and expression of any input-output function depend on the number and location of the samplers.



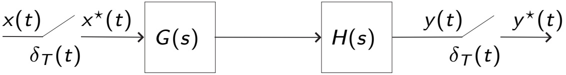
$\Rightarrow \quad Y(z) = G(z)X(z)$

$\Rightarrow$ $Y(z) = Z(G(s)X(s))$
The transfer function is not defined

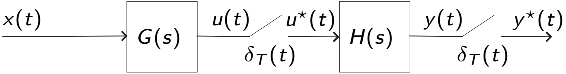
The pulse transfer function - Block diagrams (2/4)



$\Rightarrow Y(z) = G(z)H(z)X(z)$



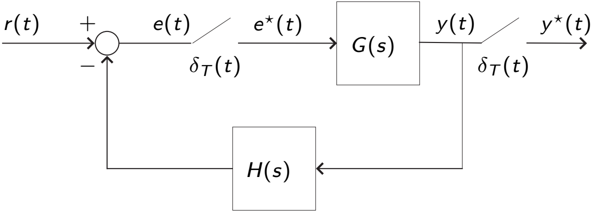
$\Rightarrow Y(z) = Z(G(s)H(s))X(z)$



$\Rightarrow$ $Y(z)=Z(H(s))Z(G(s)X(s))$
The transfer function is not defined

For ease of notation $Z(G(s)X(s)) = Z(GX(s)) = GX(z)$.

The pulse transfer function - Block diagrams (3/4)



$$\begin{aligned} E(s) &= R(s) - H(s)Y(s) \\ Y(s) &= G(s)E^\star(s) \end{aligned} \quad \Rightarrow \quad \begin{aligned} E(s) &= R(s) - H(s)G(s)E^\star(s) \\ Y(s) &= G(s)E^\star(s) \end{aligned}$$

$$\Downarrow$$

$$Y^\star(s) = \frac{G^\star(s)}{1 + [H(s)G(s)]^\star} R^\star(s) \quad \Leftarrow \quad \begin{aligned} E^\star(s) &= R^\star(s) - [H(s)G(s)]^\star E^\star(s) \\ Y^\star(s) &= G^\star(s)E^\star(s) \end{aligned}$$

$$\Downarrow$$

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

The pulse transfer function - Block diagrams (4/4)

To determine the transfer function of general interconnected systems the following systematic procedure can be used.

(1) Introduce a variable, with *name* $X_i(s)$, at the input of each sampler and a variable, with *name* $X_i^\star(s)$, at the output of each sampler.

(2) Express the input variables $X_i(s)$ of the samplers and the output variables of the system in terms of the output $X_i^\star(s)$ of the samplers and of the input variables of the system.

(3) Trasform the equations in (2) in terms of sampled quantities.

(4) Solve the equations in (3) to derive a relation between sampled input and output signals.

(5) Express the result obtained in (4) in terms of z-transforms.

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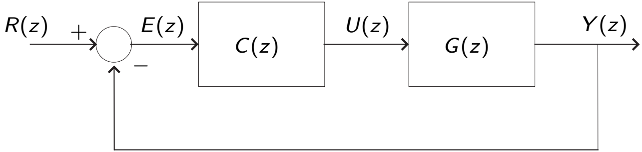
Stability and performance

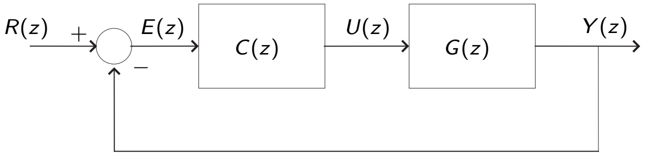


Consider a unity feedback discrete-time system.

Assume

- $G(z) = \dfrac{N_G(z)}{D_G(z)}$ and $C(z) = \dfrac{N_C(z)}{D_C(z)}$, with $N_G(z)$, $D_G(z)$, $N_C(z)$, $D_C(z)$ polynomials;
- $D_G(z)$ and $D_C(z)$ are monic;
- the relative degree r_G of $G(z)$ (i.e. $r_G = \deg D_G(z) - \deg N_G(z)$) is non-negative;
- the relative degree r_C of $C(z)$ (i.e. $r_C = \deg D_C(z) - \deg N_C(z)$) is non-negative;
- $r_G + r_C \geq 1$;
- the polynomials $N_G(z)N_C(z)$ and $D_G(z)D_C(z)$ are co-prime (i.e. they have no common root).

Stability and performance



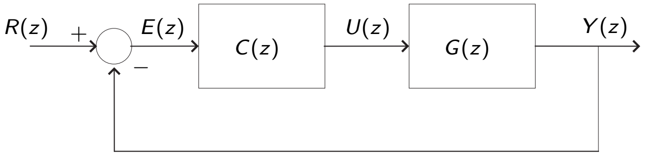
Stability and performance properties of the feedback system are properties of the characteristic polynomial of the system, i.e. of the polynomial

$$D_G(z)D_C(z) + N_G(z)N_C(z) = \mathrm{num}(1 + C(z)G(z))$$

and of the closed-loop transfer functions that can be obtained considering, for example, the input $R(z)$ and the outputs $Y(z)$, $E(z)$, $U(z)$, i.e.

$$\frac{Y(z)}{R(z)} = \frac{C(z)G(z)}{1 + C(z)G(z)}, \qquad \frac{E(z)}{R(z)} = \frac{1}{1 + C(z)G(z)}, \qquad \frac{U(z)}{R(z)} = \frac{C(z)}{1 + C(z)G(z)}.$$

Stability (1/5)



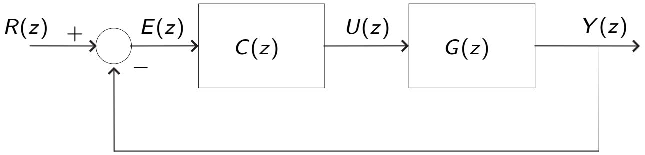
Let

$$1 + C(z)G(z) = \frac{P(z)}{D(z)} = \frac{a_0 z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n}{b_0 z^n + b_1 z^{n-1} + \cdots b_{n-1} z + b_n}.$$

If the underlying linear systems are reachable (stabilizable) and observable (detectable) then the stability properties of the system are related to the location of the zeros of $1 + C(z)G(z)$, i.e. to the location of the roots of $P(z)$.

Note that by the stated assumptions $P(z)$ is monic, i.e. $a_0 = 1$.

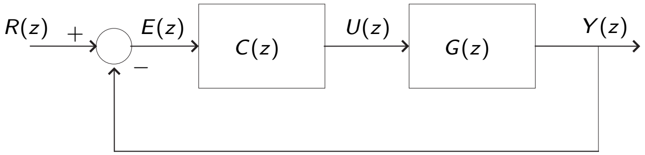
Stability (2/5)



$$P(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$$

- The system is asymptotically stable if and only if all roots of $P(z)$ are in D^-.
- The system is stable if all roots of $P(z)$ are in $D^- \cup D_0$ and the roots in D^0 are **simple**. (D^0 denotes the boundary of the unity disk.)
- The system is unstable if $P(z)$ has roots in D^+ or **multiple** roots on D^0. (D^+ denotes the exterior of the unity disk.)

Stability (2/5)

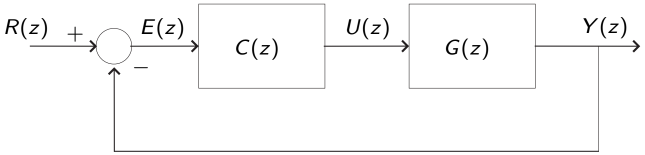


$$P(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$$

- The system is unstable if $P(z)$ has roots in D^+ or **multiple** roots on D^0. (D^+ denotes the exterior of the unity disk.)

Be careful: A "simple root" is a root with multiplicity 1. A "multiple root" is a root with multiplicity larger than 1, also called a "repeated root". For example, in the equation $(z-1)^2 = 0$, 1 is a multiple root because it has multiplicity 2. Thus, examples of polynomials with multiple roots in D^0 are, for instance, $(z^2+1)^2 = 0$ [roots: $+i, +i, -i, -i$] or $z^3 = 0$ [roots: $0, 0, 0$]. Note that "multiple" DOES NOT mean "more than one". For example $z(z^2+1) = 0$ has more than one root but it does not have multiple roots in D^0 because it has 3 simple roots $(0, +i, -i)$ in D^0 (that is, all the roots are different and so have multiplicity 1).

Stability (2/5)



$$P(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$$

- The system is asymptotically stable if and only if all roots of $P(z)$ are in D^-.

Asymptotic stability implies that

- the impulse response of any closed-loop transfer function is bounded and converges to zero for $k \to \infty$;
- any bounded input sequence yields a bounded output sequence.

Stability (3/5)

Stability of the system can be assessed with various methods.

- Compute the roots of $P(z)$.
- Develop algorithms that locate the roots of $P(z)$ with respect to D^0 without computing the roots.
- Use graphical methods exploiting the frequency response of the system.

The first method is computationally expensive (may be numerically ill-posed) and yields information which is not needed.

The second method is similar in spirit to Routh criterion, which however locates the roots of a polynomial with respect to the imaginary axis.

The third method requires to sketch the graph of the frequency response of the open-loop transfer function. Since the frequency response is a transcendental function it may be difficult to obtain such a graph.

Stability (4/5)

Consider the polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$$

and the *bilinear* transformation

$$z = \frac{1+w}{1-w} \qquad\qquad w = \frac{z-1}{z+1}.$$

The bilinear transformation maps the set $|z| = 1$ into $Re(w) = 0$, the set $|z| < 1$ into $Re(w) < 0$, and the set $|z| > 1$ into $Re(w) > 0$.

Stability (4/5)

Consider the polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$$

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$$z = \frac{1+w}{1-w} \qquad\qquad w = \frac{z-1}{z+1}.$$

The bilinear transformation maps the set $|z| = 1$ into $Re(w) = 0$, the set $|z| < 1$ into $Re(w) < 0$, and the set $|z| > 1$ into $Re(w) > 0$.

Proof. Let $w = \sigma + j\omega$ and note that

$$|z| = \left|\frac{1+w}{1-w}\right| = \left|\frac{1+\sigma+j\omega}{1-\sigma-j\omega}\right| = \frac{\sqrt{(1+\sigma)^2+\omega^2}}{\sqrt{(1-\sigma)^2+\omega^2}}.$$

Hence

$$\begin{aligned}
|z| = 1 &\Rightarrow (1+\sigma)^2+\omega^2 = (1-\sigma)^2+\omega^2 \Rightarrow \sigma = 0 \\
|z| < 1 &\Rightarrow (1+\sigma)^2+\omega^2 < (1-\sigma)^2+\omega^2 \Rightarrow \sigma < 0 \\
|z| > 1 &\Rightarrow (1+\sigma)^2+\omega^2 > (1-\sigma)^2+\omega^2 \Rightarrow \sigma > 0
\end{aligned}$$

Consider the polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$$

and the *bilinear* transformation

$$z = \frac{1+w}{1-w} \qquad\qquad w = \frac{z-1}{z+1}.$$

The bilinear transformation maps the set $|z| = 1$ into $Re(w) = 0$, the set $|z| < 1$ into $Re(w) < 0$, and the set $|z| > 1$ into $Re(w) > 0$.

The bilinear transformation is not the only one with the above properties. The transformation

$$z = \frac{1 + \frac{wT}{2}}{1 - \frac{wT}{2}} \qquad\qquad w = \frac{2}{T}\frac{z-1}{z+1}$$

with $T > 0$ has the same properties.

Stability (4/5)

Consider the polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$$

and the *bilinear* transformation

$$z = \frac{1+w}{1-w} \qquad\qquad w = \frac{z-1}{z+1}.$$

The bilinear transformation maps the set $|z| = 1$ into $Re(w) = 0$, the set $|z| < 1$ into $Re(w) < 0$, and the set $|z| > 1$ into $Re(w) > 0$.

The w-plane is *similar* to the s-plane, but it is not equivalent to it. The mapping between z and s is given by

$$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} \approx \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}} = \frac{1+w}{1-w} \qquad\qquad w = \frac{sT}{2}$$

hence the mapping between z and w is an approximation of the mapping between z and s.

Stability (4/5)

Consider the polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$$

and the *bilinear* transformation

$$z = \frac{1+w}{1-w} \qquad\qquad w = \frac{z-1}{z+1}.$$

The bilinear transformation yields

$$\tilde{P}(w) = P\left(\frac{1+w}{1-w}\right) = \left(\frac{1+w}{1-w}\right)^n + a_1\left(\frac{1+w}{1-w}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1+w}{1-w}\right) + a_n.$$

$$\Downarrow$$

The roots of $P(z)$ are in a one-to-one correspondence
with the zeros of the rational function $\tilde{P}(w)$.

Stability (4/5)

Consider the polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$$

and the *bilinear* transformation

$$z = \frac{1+w}{1-w} \qquad\qquad w = \frac{z-1}{z+1}.$$

The bilinear transformation yields

$$\tilde{P}(w) = P\left(\frac{1+w}{1-w}\right) = \left(\frac{1+w}{1-w}\right)^n + a_1\left(\frac{1+w}{1-w}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1+w}{1-w}\right) + a_n.$$

$$\Downarrow$$

The roots of $P(z)$ are in a one-to-one correspondence with the roots of the polynomial $Q(w) = \mathrm{num}\tilde{P}(w)$.

Stability (4/5)

Consider the polynomials

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \qquad Q(w) = q_0 w^n + q_1 w^{n-1} + \cdots + q_{n-1} w + q_n.$$

Note that $q_0 = 0$ if and only if $P(-1) = 0$, i.e. $Q(w)$ has degree smaller than n if and only if $P(z)$ has roots for $z = -1$.

$P(z)$ has all roots in D^- if and only if $Q(w)$ has all roots in $\mathbb{C}^-$.

The number of roots of $P(z)$ in D^- (D^+, resp.) coincides with the number of roots of $Q(w)$ in C^- (C^+, resp.).

If $q_0 \neq 0$, the number of roots of $P(z)$ in D^0 coincides with the number of roots of $Q(w)$ in C^0.

Stability of a discrete-time system can be assessed with Routh test.

Routh Theorem - Recall – Stability (4/5)

Given: $\varphi(s) = \varphi_0 s^n + \varphi_1 s^{n-1} + \cdots + \varphi_{n-1} s + \varphi_n$ we build the table

$$
n+1 \text{ rows} \left\{
\begin{array}{c|cccc}
n & \varphi_0 & \varphi_2 & \varphi_4 & \cdots \\
n-1 & \varphi_1 & \varphi_3 & \varphi_5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots \\
i & h_1 & h_2 & h_3 & \cdots \\
i-1 & k_1 & k_2 & k_3 & \cdots \\
i-2 & l_1 & l_2 & l_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots \\
0 & & & &
\end{array}
\right.
$$

To compute the next line of coefficients

$$
l_1 = -\frac{1}{k_1} \det \begin{bmatrix} h_1 & h_2 \\ k_1 & k_2 \end{bmatrix} \qquad l_2 = -\frac{1}{k_1} \det \begin{bmatrix} h_1 & h_3 \\ k_1 & k_3 \end{bmatrix}
$$

and in general

$$
l_j = -\frac{1}{k_1} \det \begin{bmatrix} h_1 & h_{j+1} \\ k_1 & k_{j+1} \end{bmatrix}
$$

Routh Theorem - Recall – Stability (4/5)

Routh-Hurwitz theorem

If the Routh table is well-defined (that is, if the elements of the first column are not zero except at most the last one), then:

- The number of roots with positive real part is equal to the number of sign-changes in the first column of the Routh table.
- The number of roots with negative real part is equal to the number of sign-permanences in the first column of the Routh table.

Stability (5/5)

Consider a discrete-time system with transfer function $G(z)$. The function

$$G(e^{j\omega T}) \qquad 0 \le \omega \le \frac{\pi}{T}$$

is the frequency response of the system.

- The frequency response is a trascendental function.
- The frequency response is defined for $0 \le \omega \le \frac{\pi}{T}$, since it is periodic of period $\omega_s = \frac{2\pi}{T}$ and for negative ω takes complex conjugate values, i.e.

$$G(e^{j(\omega + k\omega_s)T}) = G(e^{j\omega T}) \qquad G(e^{j(-\omega)T}) = \bar{G}(e^{j\omega T}).$$

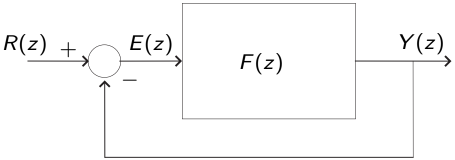
- If $G(z)$ has all poles inside D^-, i.e. the system is asymptotically stable, then the frequency response allows to describe the steady-state response of the output of the system to a sinusoidal input:

$$x(kT) = \sin k\tilde{\omega}T \quad \Rightarrow \quad y(kT) = A\sin(k\tilde{\omega}T + \phi)$$

  where

$$A = |G(e^{j\tilde{\omega}T})| \qquad \phi = \angle G(e^{j\tilde{\omega}T}).$$

Stability (5/5)



Consider a unity feedback discrete-time system with open loop transfer function $F(z)$.

Consider the image on the complex plane of the frequency response function

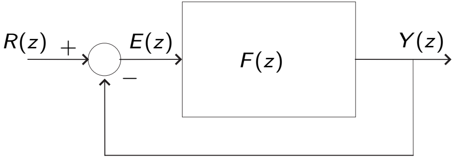
$$F(e^{j\omega T}) \qquad\qquad -\frac{\pi}{T} \leq \omega \leq \frac{\pi}{T}.$$

This image is called the Nyquist diagram of $F(z)$.

If $F(z)$ does not have poles on D^0 the Nyquist diagram is a closed curve.

If $F(z)$ has poles on D^0 the Nyquist diagram *contains* the point at infinity (i.e. it is closed at infinity).

Stability (5/5)



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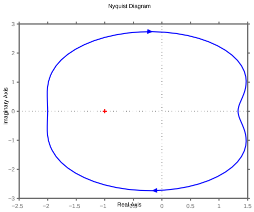

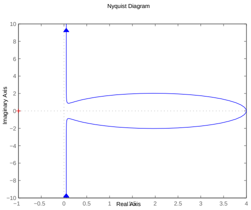

Stability (5/5)

Nyquist criterion I

Consider the function $F(z)$. Assume all poles of $F(z)$ are in D^-, with the exception of a single or double pole at $z = 1$.

The closed-loop system is asymptotically stable if and only if the Nyquist diagram of $F(z)$ does not encircle the point $-1 + j0$.

Stability (5/5)

Nyquist criterion I

Consider the function $F(z)$. Assume all poles of $F(z)$ are in D^-, with the exception of a single or double pole at $z = 1$.

The closed-loop system is asymptotically stable if and only if the Nyquist diagram of $F(z)$ does not encircle the point $-1 + j0$.

Nyquist criterion II

Consider the function $F(z)$. Assume $F(z)$ does not have poles on D^0, with the exception of a single or double pole at $z = 1$.

The closed-loop system is asymptotically stable if and only if the number of anti-clock-wise encirclements of $-1 + j0$ of the Nyquist diagram of $F(z)$ equals the number of poles of $F(z)$ in D^+.

Consider the discrete-time transfer function

$$F(z) = \frac{z}{(z-1)(z-1/2)}.$$

The frequency response function is

$$F(e^{j\omega T}) = \frac{e^{j\omega T}}{(e^{j\omega T}-1)(e^{j\omega T}-1/2)}$$

Note that

$$F(e^{j\omega T}) \approx \begin{cases} \frac{2}{j\omega T} & \text{if } |\omega| \ll 1 \\ -1/3 & \text{if } |\omega| \approx \dfrac{\pi}{T} \end{cases}$$

and that

$$F(e^{j\omega T}) = -\frac{1}{\alpha},$$

for some $\alpha > 0$, only if $\omega = \pm\frac{\pi}{T}$, yielding $\alpha = 3$.

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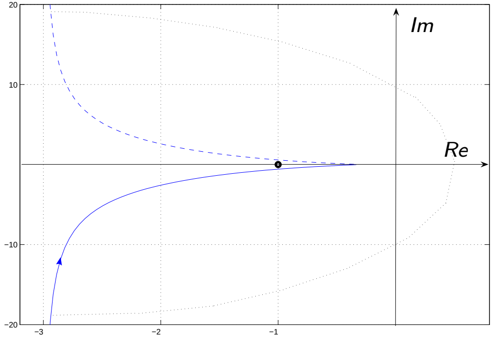
Stability (5/5) – Example

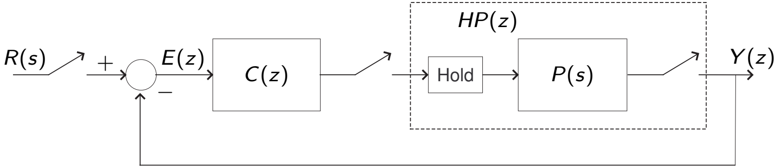
Consider the discrete-time transfer function

$$F(z) = \frac{z}{(z-1)(z-1/2)}.$$



The closed-loop system is asymptotically stable. Moreover, the closed-loop system with open-loop transfer function $kF(z)$ is asymptotically stable for all $k \in (0, 3)$.

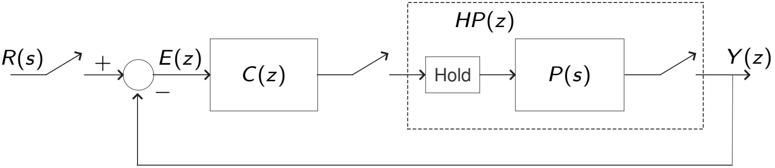
Performance (1/5)



Consider a computer controlled system, i.e. a continuous-time system, with transfer function $P(s)$, interconnected by means of sampling and hold devices, to a discrete-time controller, with transfer function $C(z)$.

The performance of the control system can be quantified in terms of the following indicators.

Performance (1/5)



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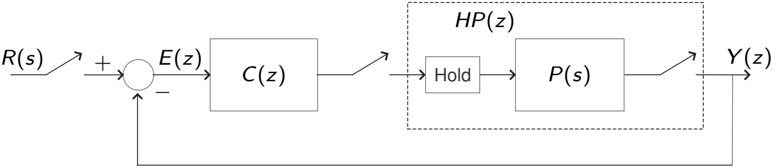
The performance of the control system can be quantified in terms of the following indicators.

- Nominal stability.
  The closed-loop system should be asymptotically stable.

Performance (1/5)

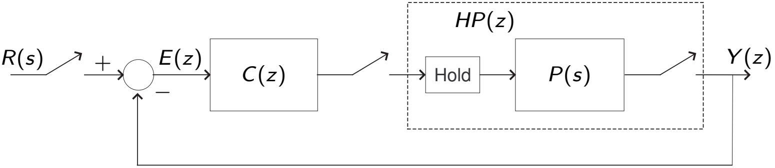


Consider a computer controlled system, i.e. a continuous-time system, with transfer function $P(s)$, interconnected by means of sampling and hold devices, to a discrete-time controller, with transfer function $C(z)$.

The performance of the control system can be quantified in terms of the following indicators.

- Nominal stability.
- Robust stability.
  The stability properties of the closed-loop system should be preserved in the presence of (small) perturbations on the plant. (The closed-loop system should possess certain *stability margins*.)

Performance (1/5)

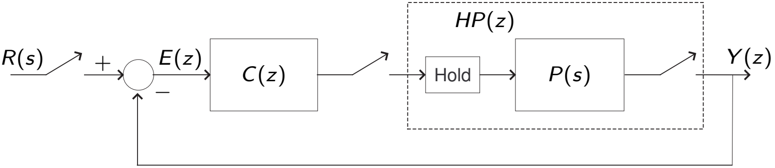


Consider a computer controlled system, i.e. a continuous-time system, with transfer function $P(s)$, interconnected by means of sampling and hold devices, to a discrete-time controller, with transfer function $C(z)$.

The performance of the control system can be quantified in terms of the following indicators.

- Nominal stability.
- Robust stability.
- Steady-state accuracy.
  The steady-state response of the closed-loop system to classes of input signals (references, disturbances) should possess specific properties.

Performance (1/5)

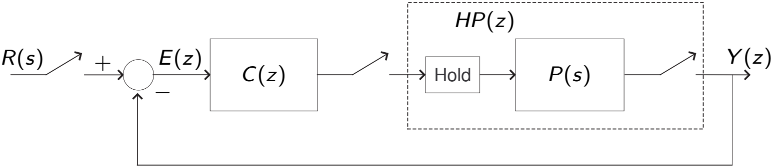


Consider a computer controlled system, i.e. a continuous-time system, with transfer function $P(s)$, interconnected by means of sampling and hold devices, to a discrete-time controller, with transfer function $C(z)$.

The performance of the control system can be quantified in terms of the following indicators.

- Nominal stability.
- Robust stability.
- Steady-state accuracy.
- Transient response.
  The dynamic behaviour of the output of the closed-loop system should be within specific bounds, given in terms of overshot, settling time, ...

Performance (1/5)



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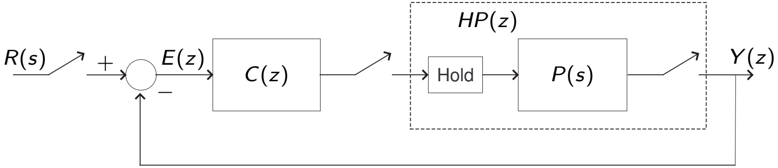
The performance of the control system can be quantified in terms of the following indicators.

- Nominal stability.
- Robust stability.
- Steady-state accuracy.
- Transient response.
- Disturbance rejection/attenuation.
  The effect of classes of disturbances on the output of the closed-loop system should be *small*.

Performance (1/5)



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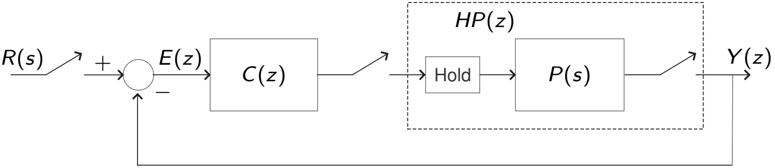
The performance of the control system can be quantified in terms of the following indicators.

- Nominal stability.
- Robust stability.
- Steady-state accuracy.
- Transient response.
- Disturbance rejection/attenuation.

Steady-state accuracy – Performance (2/5)



Consider a computer controlled system. Assume that the hold device is a zero-order hold.

The discrete-time description of the plant is given by

$$HP(z) = Z(H_0(s)P(s)) = (1 - z^{-1})Z\left(\frac{P(s)}{s}\right),$$

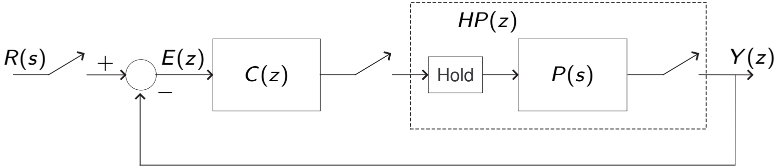
hence the open loop transfer function is

$$G(z) = C(z)HP(z)$$

and

$$E(z) = \frac{1}{1 + G(z)}R(z).$$

Steady-state accuracy – Performance (2/5)



Assume the closed-loop system is stable. Let $r(k) = 1$, i.e. $R(z) = \frac{1}{1-z^{-1}}$. Then

$$e_p = \lim_{z\to1}(1-z^{-1})E(z) = \lim_{z\to1}(1-z^{-1})\frac{1}{1+G(z)}\frac{1}{1-z^{-1}} = \frac{1}{1+k_p},$$

where

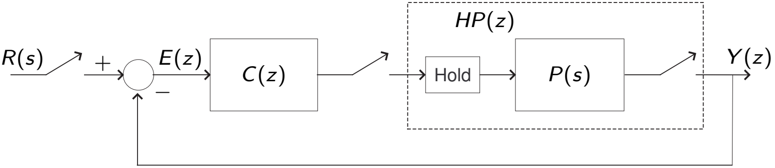
$$k_p = \lim_{z\to1} G(z) \qquad \text{(position constant)}$$

$e_p = 0$ if and only if $k_p = \infty$, i.e. $G(z)$ has at least one pole at $z = 1$.

The *type* of the feedback system is determined by the number of poles of the open-loop transfer function at $z = 1$.

The steady-state error for constant reference input signals is zero if and only if the system is of type at least 1.

Steady-state accuracy – Performance (2/5)



Assume the closed-loop system is stable. Let $r(k) = kT$, i.e. $R(z) = \frac{Tz^{-1}}{(1-z^{-1})^2}$. Then

$$e_v = \lim_{z\to1}(1-z^{-1})E(z) = \lim_{z\to1}(1-z^{-1})\frac{1}{1+G(z)}\frac{Tz^{-1}}{(1-z^{-1})^2} = \lim_{z\to1}\frac{T}{(1-z^{-1})G(z)}.$$

Let

$$k_v = \lim_{z\to1}\frac{(1-z^{-1})G(z)}{T} \qquad \text{(velocity constant)}$$

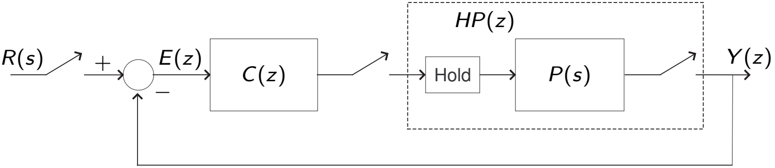
then

$$e_v = \frac{1}{k_v}.$$

e_v is finite if and only if $k_v \neq 0$, i.e. the system is of type at least 1.

$e_v = 0$ if and only if $k_v = \infty$, i.e. the system is of type at least 2.

Steady-state accuracy – Performance (2/5)



Assume the closed-loop system is stable. Let $r(k) = \frac{1}{2}(kT)^2$, i.e. $R(z) = \frac{T^2(1+z^{-1})z^{-1}}{2(1-z^{-1})^3}$. Then

$$e_a = \lim_{z\to 1}(1-z^{-1})E(z) = \lim_{z\to 1}(1-z^{-1})\frac{1}{1+G(z)}\frac{T^2(1+z^{-1})z^{-1}}{2(1-z^{-1})^3} = \lim_{z\to 1}\frac{T^2}{(1-z^{-1})^2G(z)}.$$

Let

$$k_a = \lim_{z\to 1}\frac{(1-z^{-1})^2G(z)}{T^2} \qquad \text{(acceleration constant)}$$

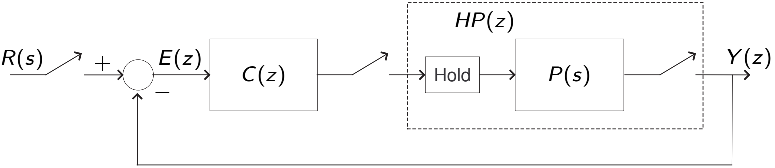
then

$$e_a = \frac{1}{k_a}.$$

e_a is finite if and only if $k_a \neq 0$, i.e. the system is of type at least 2.

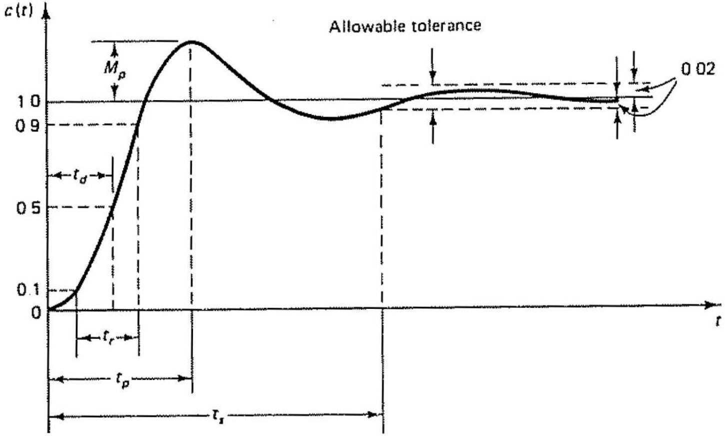
$e_a = 0$ if and only if $k_a = \infty$, i.e. the system is of type at least 3.

Steady-state accuracy – Performance (2/5)



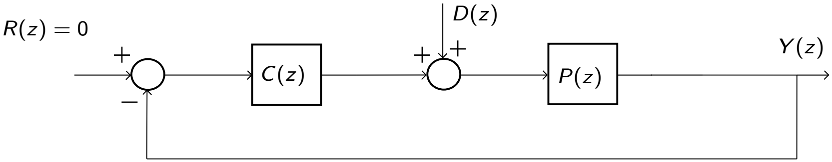
System\Input	$r(k)=1$	$r(k)=kT$	$r(k)=\frac{1}{2}(kT)^2$	$\cdots$
Type 0	$\frac{1}{1+K_p}$	∞	∞	$\cdots$
Type 1	0	$\frac{1}{K_v}$	∞	$\cdots$
Type 2	0	0	$\frac{1}{K_a}$	$\ddots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$

Transient response – Performance (3/5)



Delay time t_d - Rise time t_r - Peak time t_p - Maximum overshoot M_p - Settling time t_s

Disturbance reject./atten. – Performance (4/5)



The closed-loop pulse transfer function from $D(z)$ to $Y(z)$ is

$$\frac{Y(z)}{D(z)} = \frac{P(z)}{1 + C(z)P(z)}.$$

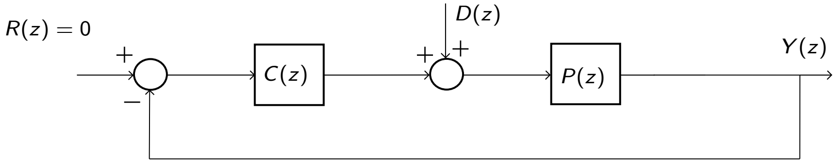
If $|C(z)P(z)| >> 1$ then we have

$$\frac{Y(z)}{D(z)} \approx \frac{1}{C(z)}$$

and the system error is

$$E(z) = R(z) - Y(z) = -\frac{1}{C(z)}D(z)$$

Disturbance reject./atten. – Performance (4/5)

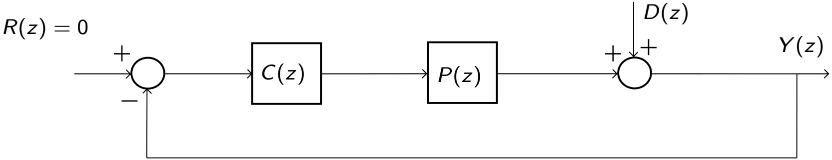


Assume that $C(z)$ has a pole at $z=1$, *i.e.* $C(z) = \dfrac{\hat{C}(z)z^{-1}}{1-z^{-1}}$ where $\hat{C}(z)$ does not have zeros at $z=1$. Let $d(k)=N$, i.e. $D(z) = \dfrac{N}{1-z^{-1}}$. Then

$$e_{ss} = \lim_{z\to 1}(1-z^{-1})E(z) = \lim_{z\to 1}(1-z^{-1})\frac{-1}{C(z)}\frac{N}{1-z^{-1}} = -\lim_{z\to 1}\frac{(1-z^{-1})N}{\hat{C}(z)z^{-1}} = 0,$$

If $R(z) \neq 0$ then a new analysis should be carried out since both the reference input and the disturbance will contribute to the steady-state error.

Disturbance reject./atten. – Performance (4/5)

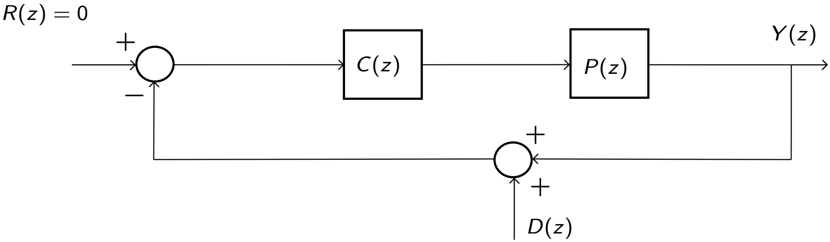


The point where the disturbance enters is very important! The closed-loop pulse transfer function from $D(z)$ to $Y(z)$ is

$$\frac{Y(z)}{D(z)} = -\frac{E(z)}{D(z)} = \frac{1}{1 + C(z)P(z)}$$

thus the gain of $C(z)P(z)$ must be made as large as possible.

Disturbance reject./atten. – Performance (4/5)



The closed-loop pulse transfer function from $D(z)$ to $Y(z)$ is

$$\frac{Y(z)}{D(z)} = -\frac{C(z)P(z)}{1 + C(z)P(z)}$$

thus the gain of $C(z)P(z)$ must be made as small as possible.

Performance (5/5)

Dynamic properties of continuous-time control systems can be expressed in terms of the location on the s-plane of the closed-loop poles, i.e. in terms of their natural frequencies and damping coefficients.

For discrete-time systems it is necessary to consider the location of the closed-loop poles in the z-plane.

The s- and z-planes are related by the equations

$$z = e^{sT} \qquad\qquad s = \frac{1}{T}\log z$$

which allows to determine dynamic properties of discrete-time control systems from dynamic properties of continuous-time control systems.

The closed-loop poles of a discrete-time dynamical system are (in general) more sensitive to parameter variations than those of continuous-time systems.

Outline

- Introduction to digital control systems
- Z-transform: definition, properties and theorems
- Sampling and reconstruction
- The pulse transfer function
- Stability and performance
- Control design (discretization, W-plane, root locus and analytical methods)
- State space approach
- Optimal control (dynamic programming and LQR)
- Some advanced topics

Control design

The design of computer controlled systems can be performed from different perspectives.

- Indirect method.
  Controller design is performed in continuous-time, for example in the Laplace domain. The continuous-time controller is then transformed into a discrete-time controller with conversion algorithms.

- Direct method.
  Controller design is performed in discrete-time. The design can be performed
  - in the frequency domain (w-plane design);
  - in the z-domain (root locus design);
  - with analytical methods.

- Standard regulators.
  Controller design is performed in discrete-time. The structure of the controller is fixed (e.g. PID type), and the controller has to be tuned.

Control design – Warning on pole-zero cancellations

Illicit cancellations (continuous-time, discrete-time)

Unstable poles must not be canceled with unstable zeros. An unstable pole-zero cancellation is illicit!

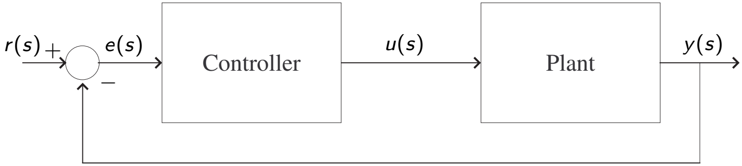
In continuous-time, we cannot cancel poles and zeros which do not lie on the complex left half-plane. In discrete-time, we cannot cancel poles and zeros which do not lie inside the unit circle.

The reason for this is that, even though the pole will not show up in the response to the input, it will still appear as a result of any initial condition of the system, or due to additional inputs entering the system (such as disturbances). If the pole and zero are in the unstable region of the plane, the system response might blow up due to these initial conditions or disturbances, even though the input of the system is bounded.

Approximate cancellations (discrete-time)

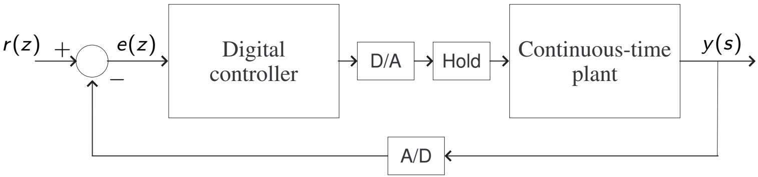
Remember that numerous approximations have been performed to obtain $HP(z)$. The poles and zeros of $HP(z)$ are approximations! When we cancel these poles or zeros we do not really cancel them but place an additional zero or pole close to these approximated poles. Hence, the final closed-loop system should be tested to verify that the approximations do not disrupt the design. This is specially true for poles and zeros close to the stability margin.

Control design – Indirect method



Consider a unity feedback continuous-time system, with plant $P(s)$ and controller $C(s)$.

Suppose the controller $C(s)$ is fixed and consider the problem of designing a discrete-time controller $C(z)$ such that the closed-loop system, with such controller and the sampler and hold devices, has performance *as close as possible* to the performance of the continuous-time closed-loop system.



Control design – Indirect method

Consider a unity feedback continuous-time system, with plant $P(s)$ and controller $C(s)$.

Suppose the controller $C(s)$ is fixed and consider the problem of designing a discrete-time controller $C(z)$ such that the closed-loop system, with such controller and the sampler and hold devices, has performance *as close as possible* to the performance of the continuous-time closed-loop system.

It is obvious that the use of a discrete-time controller, obtained by means of some *discretization* algorithm, modifies the performance of the closed-loop system.

These modifications depend upon the discretization algorithm and the sampling time.

In general, the discretization algorithm preserves some of the following properties:

- number of poles and/or zeros;
- invariance of the impulse/step response;
- DC-gain;
- stability margins and/or bandwidth.

Control design – Indirect method – Design steps

Once the continuous-time controller has been selected, the design procedure is composed of the following steps.

- Definition of the sampling time T.
- Preliminary analysis of the effect of the digital implementation on stability and performance, and redesign, if necessary, of $C(s)$.

Control design – Indirect method – Design steps

Once the continuous-time controller has been selected, the design procedure is composed of the following steps.

- Definition of the sampling time T.

- Preliminary analysis of the effect of the digital implementation on stability and performance, and redesign, if necessary, of $C(s)$.

  The effect of the digital implementation can be evaluated considering an approximation of the hold. For a ZOH we could use

  $$H_0(s) \approx \frac{T}{\frac{T}{2}s+1} \quad \text{(Padè approximation)} \qquad H_0(s) \approx Te^{-sT/2}.$$

  Note that a factor T should be removed since it is compensated for by the gain $1/T$ of the sampler, e.g. the effect of the hold should be approximated by $\frac{1}{\frac{T}{2}s+1}$.

Control design – Indirect method – Design steps

Once the continuous-time controller has been selected, the design procedure is composed of the following steps.

- Definition of the sampling time T.
- Preliminary analysis of the effect of the digital implementation on stability and performance, and redesign, if necessary, of $C(s)$.
- Discretization of $C(s)$.

Control design – Indirect method – Design steps

Once the continuous-time controller has been selected, the design procedure is composed of the following steps.

- Definition of the sampling time T.
- Preliminary analysis of the effect of the digital implementation on stability and performance, and redesign, if necessary, of $C(s)$.
- Discretization of $C(s)$.
- Analysis, a-posteriori, of the dynamic behaviour of the closed-loop system.

  It is necessary to obtain the discrete-time equivalent description of the continuous-time plant, and to analise the discrete-time closed-loop system.

Control design – Indirect method – Selection of T

The sampling time T has to be selected on the basis of the following considerations.

- If the frequency response of $C(s)$ is of low-pass type, ω_s should be significantly larger than the cut-off frequency.

- The sampling time should be consistent with the dynamic properties of the continuous-time closed-loop system.

  If the continuous-time closed-loop system is required to have a settling time t_s (e.g. for the step response) then

  $$T \in \left[\frac{t_s}{10}, \frac{t_s}{4}\right]$$

  is a *good* selection.

  If the continuous-time closed-loop system has dominant poles with natural frequencies ω_n, then

  $$T \in \left[\frac{1}{10}\frac{2\pi}{\omega_n}, \frac{1}{4}\frac{2\pi}{\omega_n}\right]$$

  is a *good* selection.

Control design – Indirect method – Discretization

The process of discretization is used to obtain a discrete-time transfer function $C(z)$ from the continuous-time transfer function $C(s)$.

- Backward difference (not stability preserving).

$$C(z) = C(s)\big|_{s=\frac{1-z^{-1}}{T}}$$

- Forward difference (not stability preserving).

$$C(z) = C(s)\big|_{s=\frac{z-1}{T}}$$

- Bilinear or Tustin transformation (stability preserving).

$$C(z) = C(s)\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

- Bilinear or Tustin transformation with pre-warping (stability preserving).

$$C(z) = C(s)\big|_{s=\frac{\omega_1}{\tan\frac{\omega_1 T}{2}}\frac{1-z^{-1}}{1+z^{-1}}}$$

Control design – Indirect method – Discretization

- Impulse response invariance (stability preserving).

$$C(z) = Z(\mathcal{L}^{-1}(C(s)))$$

- Step response invariance/zero-order hold equivalent (stability preserving).

$$C(z) = (1 - z^{-1}) Z\left(\mathcal{L}^{-1}\left(\frac{C(s)}{s}\right)\right)$$

- Pole-zero correspondance (stability preserving).

$$s + a \longrightarrow 1 - e^{-aT} z^{-1}$$

Zeroes at $s = \infty$ are mapped into zeroes at $z = -1$, and the gain at $s = 0$ ($s = \infty$) is matched at $z = 1$ ($z = -1$), for low-pass (high-pass) $C(s)$. For example

$$C(s) = \frac{a}{s + a} \longrightarrow C(z) = k \frac{a(1 + z^{-1})}{1 - e^{-aT} z^{-1}}$$

with k such that $C(s)|_{s=0} = C(z)|_{z=1}$, i.e.

$$k = \frac{1 - e^{-aT}}{2a}.$$

The bilinear transformation and its inverse, namely

$$z = \frac{1 + \frac{wT}{2}}{1 - \frac{wT}{2}} \qquad \text{and} \qquad w = \frac{2}{T}\frac{z-1}{z+1},$$

relate the z-plane to an auxiliary plane, the w-plane, which approximates the s-plane.

Using the variable w as the Laplace variable s, it is possible to design control laws with classical (frequency-domain) methods.

The design procedure yields a control law $C(w)$ which has then to be transformed, using the inverse of the bilinear transformation, into a discrete-time controller $C(z)$.

The bilinear transformation preserves the position and velocity constants, i.e.

$$k_p = \lim_{z\to 1} G(z) = \lim_{w\to 0} G(w) \qquad k_v = \lim_{z\to 1} \frac{(1 - z^{-1})G(z)}{T} = \lim_{w\to 0} wG(w),$$

hence steady-state accuracy for step and ramp reference inputs is preserved.

Control design – Direct method – w-plane design – Design steps

The design procedure is composed of the following steps.

- Definition of the sampling time T.

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- Definition of the sampling time T.

  The sampling time may be computed on the basis of the desired dynamic response of the closed-loop system or as a fraction (1/2 to 1/10) of the fastest time constant of the plant.

Control design – Direct method – w-plane design – Design steps

The design procedure is composed of the following steps.

- Definition of the sampling time T.

- Computation of the discrete-time equivalent transfer function $HP(z)$ of the system to be controlled connected to the hold.

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- Definition of the sampling time T.
- Computation of the discrete-time equivalent transfer function $HP(z)$ of the system to be controlled connected to the hold.
- Transformation of $HP(z)$ into $HP(w)$.

  Note that $HP(w)$ has, in general, relative degree zero, even if the relative degree of $HP(s)$ is positive, and may be non-minimum phase, even if $HP(s)$ is not.

The design procedure is composed of the following steps.

- Definition of the sampling time T.
- Computation of the discrete-time equivalent transfer function $HP(z)$ of the system to be controlled connected to the hold.
- Transformation of $HP(z)$ into $HP(w)$.
- Determine the gain K to satisfy the requirement on a given error constant.

Control design – Direct method – w-plane design – Design steps

The design procedure is composed of the following steps.

- Definition of the sampling time T.
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- Transformation of $HP(z)$ into $HP(w)$.
- Determine the gain K to satisfy the requirement on a given error constant.
- Design, using continuous-time (frequency-domain) methods, of the controller $C(w)$ for $KHP(w)$.

Control design – Direct method – w-plane design – Design steps

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- Transformation of $C(w)$ into $C(z)$.
- Analysis, a-posteriori, of the dynamic behaviour of the closed-loop system.

Control design – Direct method – w-plane design – Example

Consider a plant $P(s) = \frac{K}{s(s+1)}$ interconnected to a compensator $C(z)$ by means of a zero-order hold. Design a digital controller in the w plane such that the phase margin is 50° and the static velocity constant K_v is 2 $\sec^{-1}$. Assume the sampling time is $T = 0.2$.

$$HP(z) = Z\left(\frac{1-e^{-0.2s}}{s}\frac{K}{s(s+1)}\right) = (1-z^{-1})Z\left(\frac{K}{s^2(s+1)}\right) = \frac{K(0.01873z + 0.01752)}{(z^2 - 1.8187z + 0.8187)}$$

We transform the pulse transfer function into the w domain.

$$HP(w) = \frac{K(0.01873\left(\frac{1+0.1w}{1-0.1w}\right) + 0.01752)}{(\left(\frac{1+0.1w}{1-0.1w}\right)^2 - 1.8187\left(\frac{1+0.1w}{1-0.1w}\right) + 0.8187)} = \frac{K(-0.000333w^2 - 0.09633w + 0.9966)}{w^2 + 0.9969w}$$

We try with a phase-lead compensator

$$C(w) = \frac{1+\tau w}{1+\frac{\tau}{m}w}$$

Control design – Direct method – w-plane design – Example

Consider a plant $P(s) = \frac{K}{s(s+1)}$ interconnected to a compensator $C(z)$ by means of a zero-order hold. Design a digital controller in the w plane such that the phase margin is 50° and the static velocity constant K_v is 2 sec^{-1}. Assume the sampling time is $T = 0.2$.

$$C(w) = \frac{1 + \tau w}{1 + \frac{\tau}{m} w}$$

The static velocity constant K_v is given by

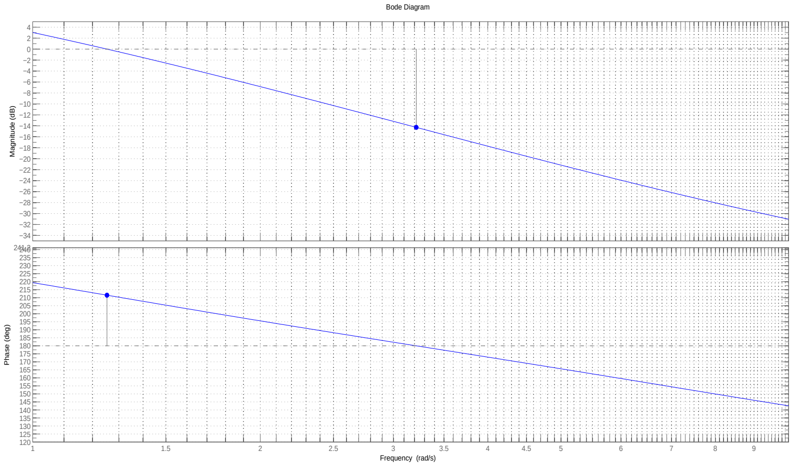
$$2 := K_v = \lim_{w \to 0} wC(w)HP(w) = \frac{0.9966Kw}{0.9969w} \approx K$$

Thus

$$HP(w) = \frac{2(-0.000333w^2 - 0.09633w + 0.9966)}{w^2 + 0.9969w}$$

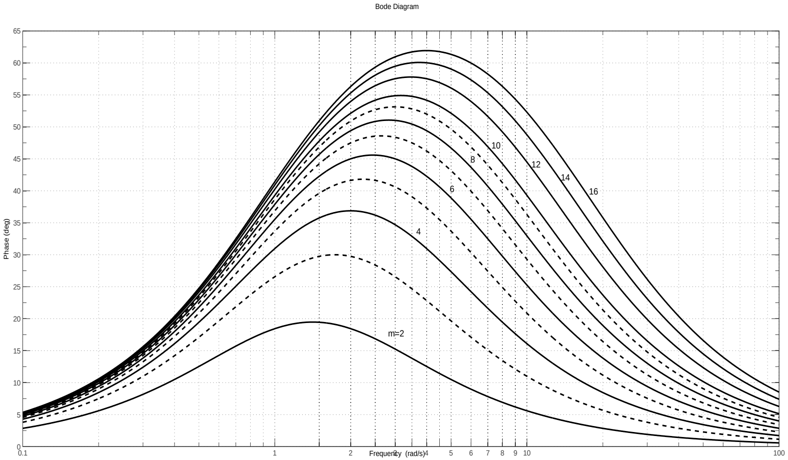
We now sketch the Bode plot.

Control design – Direct method – *w*-plane design – Example



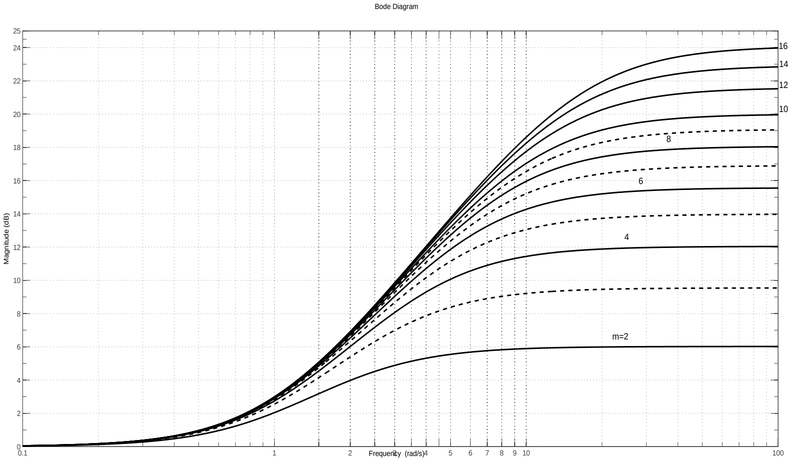
Phase margin 31.6°. Missing 18.4°

Control design – Direct method – *w*-plane design – Example



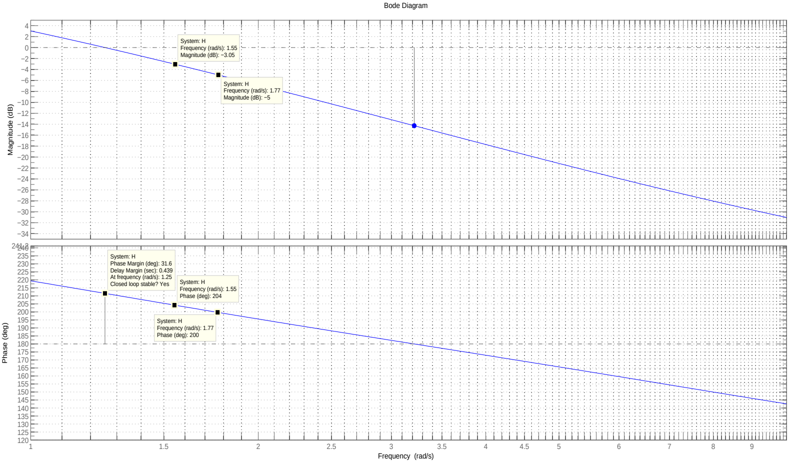
For $m = 2$, peak phase increases $\approx 20^\circ$

Control design – Direct method – w-plane design – Example



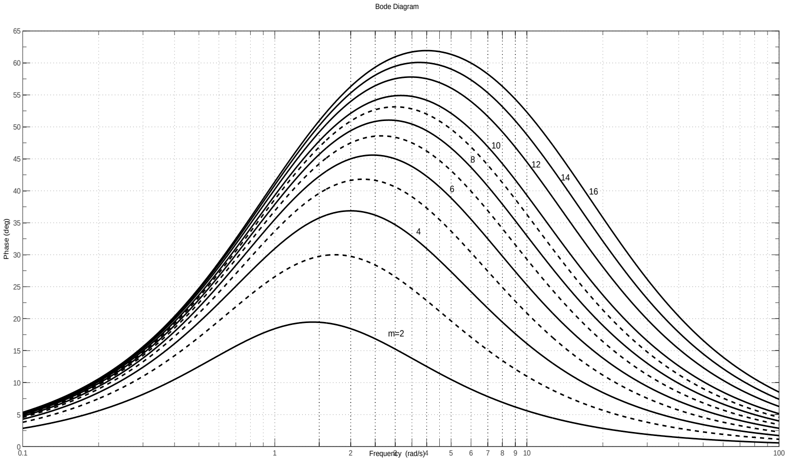
For $m = 2$, at phase peak, magnitude increases $\approx$ 3dB

Control design – Direct method – *w*-plane design – Example



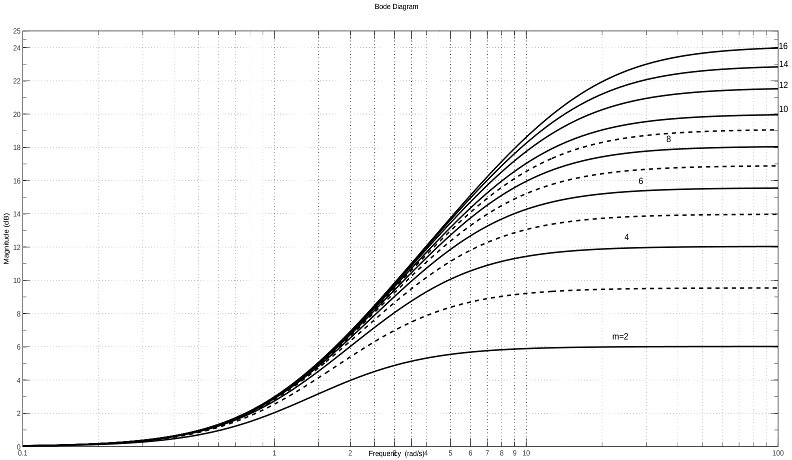
-3dB is at $\omega = 1.55$. The phase is 204°. Phase margin would be 24°. Total $24 + 20 = 44^\circ$

Control design – Direct method – *w*-plane design – Example



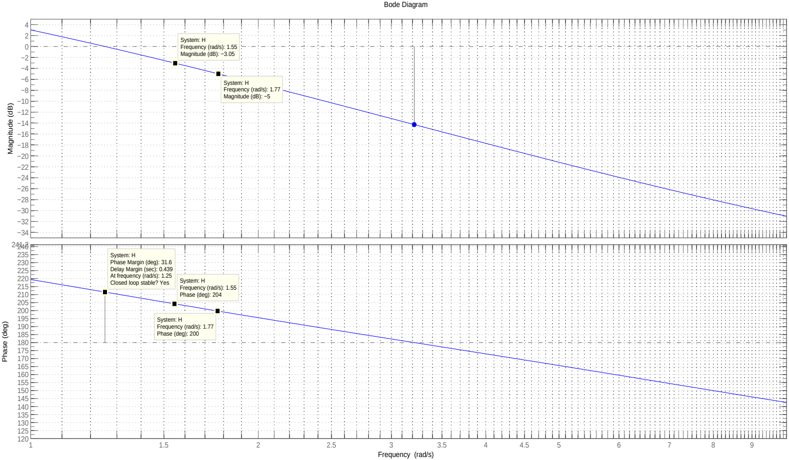
For $m = 3$, peak phase increases $\approx 30^\circ$

Control design – Direct method – *w*-plane design – Example



For $m = 3$, at phase peak, magnitude increases $\approx$ 5dB

Control design – Direct method – *w*-plane design – Example



-5dB is at $\omega = 1.77$. The phase is 200°. Phase margin would be 20°. Total $20 + 30 = 50^\circ$.

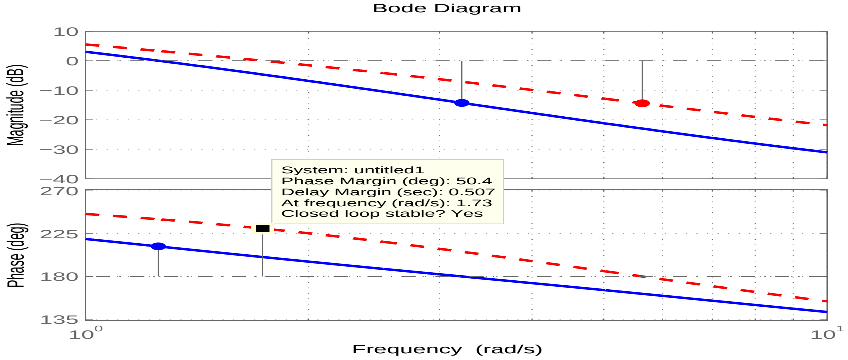
Control design – Direct method – w-plane design – Example

The phase-lead compensator

$$C(w) = \frac{1+\tau w}{1+\frac{\tau}{m}w}$$

with $m = 3$, and target frequency $\omega = 1.77$ we have $\tau = \dfrac{\sqrt{m}}{\omega} = 0.97856$. Thus,

$$C(w) = \frac{1+\tau w}{1+\frac{\tau}{m}w} = \frac{1+0.97856w}{1+0.32619w}$$



Control design – Direct method – w-plane design – Example

The phase-lead compensator

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with $m = 3$, and target frequency $\omega = 1.77$ we have $\tau = \dfrac{\sqrt{m}}{\omega} = 0.97856$. Thus,

$$C(w) = \frac{1 + \tau w}{1 + \frac{\tau}{m} w} = \frac{1 + 0.97856w}{1 + 0.32619w}$$

We can now transform the transfer function back to the z-domain

$$C(z) = \frac{1 + 0.97856\left(\frac{2}{T}\frac{z-1}{z+1}\right)}{1 + 0.32619\left(\frac{2}{T}\frac{z-1}{z+1}\right)} = \frac{2.5307z - 2.0614}{z - 0.5307}$$

The root locus can be used to design control laws for discrete-time systems.

The design procedure is composed of the following steps.

- Definition of the sampling time T.

  The sampling time may be computed on the basis of the desired dynamic response of the closed-loop system or may be assigned a-priori.

The root locus can be used to design control laws for discrete-time systems.

The design procedure is composed of the following steps.

- Definition of the sampling time T.
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The design procedure is composed of the following steps.

- Definition of the sampling time T.
- Computation of the discrete-time equivalent transfer function $HP(z)$ of the system to be controlled connected to the hold.
- Design, using the root locus, of a controller $C(z)$ which achieves the requested performance specifications.

The design is performed on the exact discrete-time model: no validation step is necessary.

Control design – Direct method – Root locus design – Rules

The characteristic equation is

$$1 + KF(z) = 1 + K\frac{N(z)}{D(z)}$$

As K changes, so do the locations of the closed-loop poles. The root locus is the locus of the poles as a function of K. To sketch the root locus follow these rules:

- $N(z)$ has roots z_i, $i = 1, \ldots, m$, $D(z)$ has roots p_i, $i = 1, \ldots, n$. The difference between n and m is the relative degree $r = n - m$.
- The locus is symmetric about the real axis.
- There are n branches of the locus, one for each closed-loop pole.
- The locus starts ($K = 0$) at poles of $F(z)$, and ends ($K \to \infty$) at zeros of $F(z)$. There are r zeros at infinity as $K \to \infty$.
- The locus exists on the real axis to the left of an odd number of poles and zeros.
- If $r > 0$ there are asymptotes of the root locus that intersect the real axis at $\sigma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{r}$, and radiate out with angles $\theta = \pm q\frac{\pi}{r}$, where $q = 1, 3, 5, \ldots$.
- Break-away or break-in points of the locus exist where $N(z)D'(z) - N'(z)D(z) = 0$ ($'$ indicates the derivative).
- Angle of departure from complex pole p_j is $\pi + \sum_{i=1}^{m} \angle(p_j - z_i) - \sum_{i=1, i\neq j}^{n} \angle(p_j - p_i)$.
- Angle of arrival at complex zero z_j is $\pi - \sum_{i=1, i\neq j}^{m} \angle(z_j - z_i) + \sum_{i=1}^{n} \angle(z_j - p_i)$.

Control design – Direct method – Root locus design – Example

Consider a unity feedback loop in which the digital control

$$C(z) = K\frac{z}{z-1}$$

is interconnected by a zero-order hold to the plant

$$P(s) = \frac{1}{s+1}.$$

Determine the root locus for $T = 0.5$, $T = 1$, and $T = 2$.

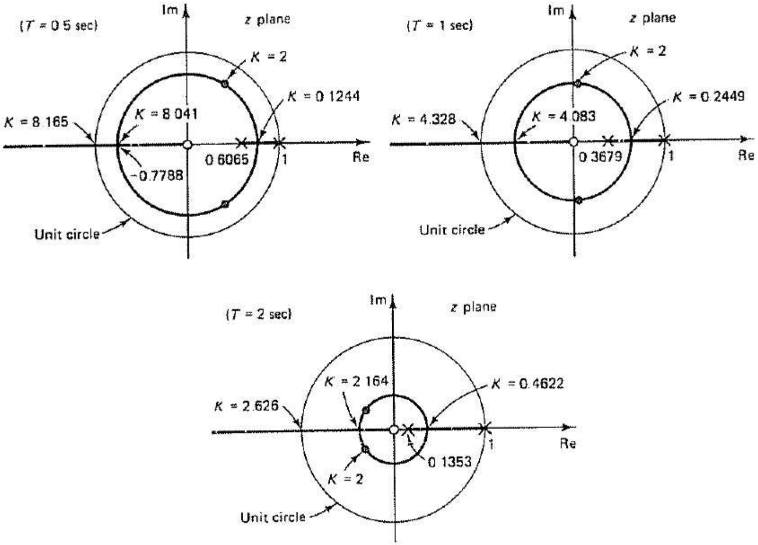
We first determine the z transform of $H_0(s)P(s)$

$$G(z) = Z[H_0(s)P(s)] = Z\left[\frac{1-e^{-Ts}}{s}\frac{1}{s+1}\right] = (1-z^{-1})Z\left[\frac{1}{s(s+1)}\right] = \frac{1-e^{-T}}{z-e^{-T}}.$$

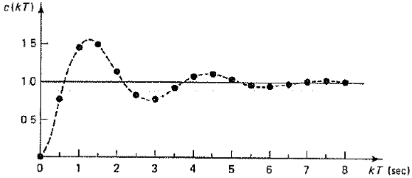
The characteristic equation is $1 + C(z)G(z) = 0$, with

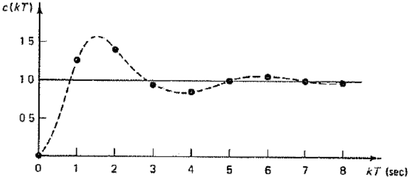
$$C(z)G(z) = \frac{Kz(1-e^{-T})}{(z-1)(z-e^{-T})}.$$

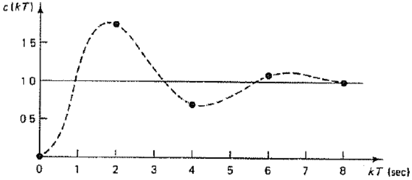
Control design – Direct method – Root locus design – Example



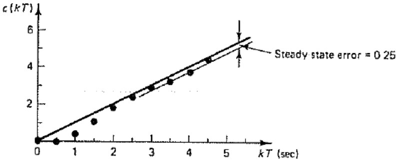
Control design – Direct method – Root locus design – Example

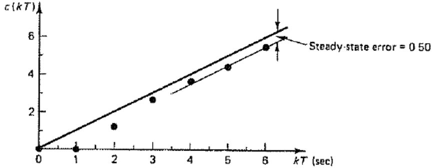


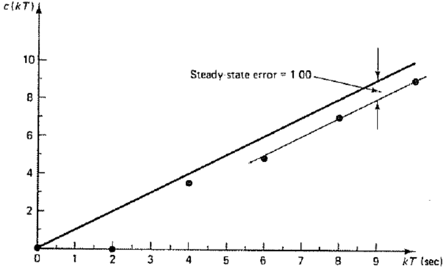




Control design – Direct method – Root locus design – Example

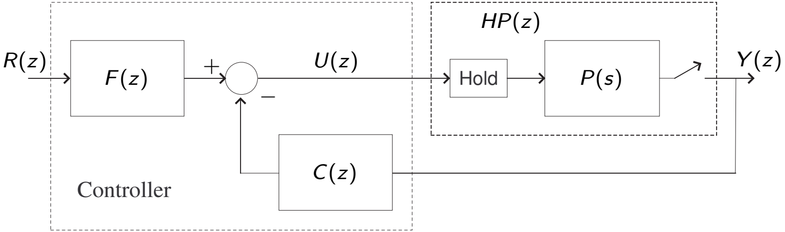






Control design – Direct method – Analytical methods

Consider a feedback control scheme with a feedback controller and a feedforward block.



Assume

$$HP(z) = \frac{B(z)}{A(z)} \qquad C(z) = \frac{S(z)}{V(z)} \qquad F(z) = \frac{T(z)}{V(z)}$$

Analytical design methods determine the polynomials $V(z)$, $S(z)$ and $T(z)$ to achieve a pre-selected closed-loop transfer function on the basis of the equation

$$\frac{Y(z)}{R(z)} = \frac{B(z)T(z)}{A(z)V(z) + B(z)S(z)}.$$

Control design – Direct method – Analytical methods

Consider a feedback control scheme with a feedback controller and a feedforward block.

Analytical design methods determine the polynomials $V(z)$, $S(z)$ and $T(z)$ to achieve a pre-selected closed-loop transfer function on the basis of the equation

$$\frac{Y(z)}{R(z)} = \frac{B(z)T(z)}{A(z)V(z) + B(z)S(z)}.$$

In the simplest version, analytical design methods are based on the assumption

$$T(z) = S(z),$$

hence the closed-loop system can be rewritten as a standard unity feedback control system, and the controller is selected

- to achieve the required steady-state accuracy;
- to assign the poles of the closed-loop transfer function, i.e. the zeros of the polynomial

$$A(z)V(z) + B(z)S(z).$$

Control design – Direct method – Analytical methods – Design steps

The design procedure is composed of the following steps.

- Selection of the desired closed-loop characteristic polynomial.
- Selection of $m \geq 0$ and definition of the polynomials

$$V(z) = z^m + v_1 z^{m-1} + \cdots + v_m \qquad S(z) = s_0 z^m + s_1 z^{m-1} + \cdots + s_m.$$

- Selection of the coefficients v_i and s_i such that
  - steady-state accuracy specification are satisfied;
  - the closed-loop characteristic polynomial coincides with the desired polynomial.

The selection of the parameter m is based on the following considerations.

- Let $n = \deg A(z)$ and assume that $A(z)$ is monic and that $HP(z)$ has relative degree larger or equal to one. Then $A(z)V(z) + B(z)S(z)$ is monic and has degree $n + m$.
- The free design parameters s_i, for $i = 0, \cdots, m$, and v_i, for $i = 1, \cdots, m$, are $2m + 1$.
- The design parameters should satisfy $n + m$ conditions (to assign the zeros of the characteristic polynomial) and p conditions to satisfy the steady-state accuracy specifications.
- The design problem has a (unique) solution if

$$2m + 1 = n + m + p \Leftrightarrow m = n + p - 1.$$

Outline

- Introduction to digital control systems
- Z-transform: definition, properties and theorems
- Sampling and reconstruction
- The pulse transfer function
- Stability and performance
- Control design (discretization, W-plane, root locus and analytical methods)
- State space approach
- Optimal control (dynamic programming and LQR)
- Some advanced topics

Discrete-time systems can be represented by nonlinear difference equations. We call the system

$$\begin{aligned} x(k+1) &= f(x(k), u(k), k) \\ y(k) &= g(x(k), u(k), k) \end{aligned}$$

nonlinear state-space representation. We focus our attention on **linear state space systems**

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

where

- $x(k)$ $n-$vector (state vector)
- $y(k)$ $m-$vector (output vector)
- $u(k)$ $r-$vector (input vector)
- A $n \times n$ matrix (state matrix)
- B $n \times r$ matrix (input matrix)
- C $m \times n$ matrix (output matrix)
- D $m \times r$ matrix (feedforward matrix)

State space – Transfer function → State space

Many techniques are available to obtain a state-space representation of a linear discrete-time system.

Consider the difference equation

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)$$

The pulse transfer function of this system is

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

State space – Transfer function → State space

The pulse transfer function of this system is

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

A state space representation of this pulse transfer function is the **controllable canonical form**:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \dots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

State space – Transfer function → State space

The pulse transfer function of this system is

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

Another state space representation of this pulse transfer function is the **observable canonical form**:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & -a_n \\ 1 & 0 & \dots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_2 \\ 0 & 0 & \dots & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

State space – Transfer function → State space

The pulse transfer function of this system is

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

- The controllable canonical form is controllable but not necessarily observable!
- The observable canonical form is observable but not necessarily controllable!

State space – Response & pulse transfer matrix

Consider the linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

The state response of the system is

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-j-1} Bu(j)$$

Proof:

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) \\ x(2) &= Ax(1) + Bu(1) = A^2 x(0) + ABu(0) + Bu(1) \\ x(3) &= Ax(2) + Bu(2) = A^3 x(0) + A^2 Bu(0) + ABu(1) + Bu(2) \\ &\vdots \\ x(k) &= A^k x(0) + \textstyle\sum_{j=0}^{k-1} A^{k-j-1} Bu(j) \end{aligned}$$

State space – Response & pulse transfer matrix

Consider the linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

The input-output pulse transfer matrix is

$$\frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D$$

Proof:

$$\begin{aligned} x(k+1) = Ax(k) + Bu(k) \quad &\rightarrow \quad zX(z) - zx(0) = AX(z) + BU(z) \\ &\rightarrow \quad X(z) = (zI - A)^{-1}zx(0) + (zI - A)^{-1}BU(z) \end{aligned}$$

Moreover

$$x(k) = \mathcal{Z}^{-1}[(zI - A)^{-1}z]x(0) + \mathcal{Z}^{-1}[(zI - A)^{-1}BU(z)]$$

and note that the following interesting mathematical relations must hold

$$\begin{aligned} A^k &= \mathcal{Z}^{-1}[(zI - A)^{-1}z] \\ \textstyle\sum_{j=0}^{k-1} A^{k-j-1}B &= \mathcal{Z}^{-1}[(zI - A)^{-1}BU(z)] \end{aligned}$$

State space – Response & pulse transfer matrix

Consider the linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

How many state-space representations there exist for a given pulse transfer function?

State space – Response & pulse transfer matrix

Consider the linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

How many state-space representations there exist for a given pulse transfer function?

Infinitely many!

In fact, the system

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) &= \hat{C}\hat{x}(k) + \hat{D}u(k) \end{aligned}$$

where $x(k) = T\hat{x}(k)$, with T **any** invertible matrix, $\hat{A} = T^{-1}AT$, $\hat{B} = T^{-1}B$, $\hat{C} = CT$ and $\hat{D} = D$, has the same transfer function of the original system.

Proof:

$$\begin{aligned} \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D} &= CT(zI - T^{-1}AT)^{-1}T^{-1}B + D \\ &= CT(zT - AT)^{-1}B + D \\ &= C(zTT^{-1} - ATT^{-1})^{-1}B + D \\ &= C(zI - A)^{-1}B + D \end{aligned}$$

State space – Sampled systems

Consider the continuous-time system

$$\begin{aligned} \dot{x}(t) &= Fx(t) + Gu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

and assume that the input is sampled and fed to a zero-order hold, *i.e.*

$$u(t) = u(kT), \qquad \text{for all } kT \leq t \leq kT + T$$

The discrete-time representation of the continuous-time systems is

$$\begin{aligned} x((k+1)T) &= Ax(kT) + Bu(kT) \\ y(kT) &= Cx(kT) + Du(kT) \end{aligned}$$

with $A = e^{FT}$ and $B = \left(\int_0^T e^{F\lambda} d\lambda\right) G$.

The discrete-time representation of the continuous-time systems is

$$\begin{aligned} x((k+1)T) &= Ax(kT) + Bu(kT) \\ y(kT) &= Cx(kT) + Du(kT) \end{aligned}$$

with $A = e^{FT}$ and $B = \left(\int_0^T e^{F\lambda} d\lambda\right) G$.

Proof: Recall

$$x(t) = e^{F(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)} Gu(\tau) d\tau$$

Then substituting $t_0 = KT$ and $t = (k+1)T$ yields

$$x((k+1)T) = e^{FT}x(kT) + \int_{kT}^{(k+1)T} e^{F((k+1)T-\tau)} Gu(\tau) d\tau$$

Let $\lambda = (k+1)T - \tau$. When $\tau = kT$ then $\lambda = T$ and when $\tau = (k+1)T$ then $\lambda = 0$. Hence $(d\lambda = -d\tau)$,

$$x((k+1)T) = e^{FT}x(kT) - \int_{T}^{0} e^{F\lambda} Gu(kT) d\lambda = e^{FT}x(kT) + \left(\int_0^T e^{F\lambda} d\lambda\right) Gu(kT)$$

- A state $\bar{x}$ is **reachable** if there exists a finite instant of time k and an input sequence $\{u(k)\}$ such that the initial state 0 of the system can be transferred to $\bar{x}$.
- The system is said to be **reachable** if all its states are reachable.
- A state $\bar{x}$ is **controllable** if there exists a finite instant of time k and an input sequence $\{u(k)\}$ such that the initial state $\bar{x}$ of the system can be transferred to 0.
- The system is said to be **controllable** if all its states are controllable.

Let $P = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$ be the **reachability matrix**.

- The system is reachable if and only if $\text{rank}\,[P] = n$.
- The system is controllable if and only if $\text{rank}\begin{bmatrix} P & A^n \end{bmatrix} = \text{rank}\,[P]$.

While in continuous-time reachability and controllability are equivalent, in discrete-time reachability implies controllability but the converse is not true!

Example: a controllable but not reachable system

$$\begin{aligned} x(k+1) &= 0 \\ y(k) &= 0. \end{aligned}$$

State space – Structural properties

Observability (determining the initial state given the input and output sequences) and **reconstructability** (determining the final state given the input and the output sequences) can be defined in a similar way.

Let $Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ be the **observability matrix**.

- The system is observable if and only if $\text{rank}\,[Q] = n$.
- The system is recontructable if and only if $\text{rank} \begin{bmatrix} Q \\ A^n \end{bmatrix} = \text{rank}\,[Q]$.

While in continuous-time observability and reconstructability are equivalent, in discrete-time observability implies reconstructability but the converse is not true!

Example: a reconstructable but not observable system

$$\begin{aligned} x(k+1) &= 0 \\ y(k) &= 0. \end{aligned}$$

The **Kalman decomposition** helps us to understand why the transfer function represents only the reachable and observable part of the system (the proof of the Kalman decomposition requires concepts of linear algebra, such as subspaces and direct sum, which are beyond the scopes of this course).

Kalman decomposition: For any linear system there exists a change of coordinates such that the matrices of the system in the new coordinates are $A_K = T^{-1}AT$, $B_K = T^{-1}B$, $C_K = CT$, $D_K = D$, where

$$A_K = \begin{bmatrix} A_{r\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{ro} & 0 & A_{24} \\ 0 & 0 & A_{\bar{r}\bar{o}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{r}o} \end{bmatrix}, \quad B_K = \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix}, \quad C_K = \begin{bmatrix} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{bmatrix}.$$

- The subsystem $\left(\begin{bmatrix} A_{r\bar{o}} & A_{12} \\ 0 & A_{ro} \end{bmatrix}, \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \end{bmatrix}, \begin{bmatrix} 0 & C_{ro} \end{bmatrix}, D \right)$ is reachable.
- The subsystem $\left(\begin{bmatrix} A_{ro} & A_{24} \\ 0 & A_{\bar{r}o} \end{bmatrix}, \begin{bmatrix} B_{ro} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{ro} & C_{\bar{r}o} \end{bmatrix}, D \right)$ is observable.
- The subsystem $(A_{ro}, B_{ro}, C_{ro}, D)$ is reachable and observable.

Hence, $C_K(zI - A_K)^{-1}B_K + D_K = C_{ro}(zI - A_{ro})^{-1}B_{ro} + D$!

State space – Design via pole placement

Given a linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

with characteristic polynomial

$$\det(zI - A) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

we want to design a state feedback control law

$$u(k) = -Kx(k)$$

such that the closed-loop system

$$x(k+1) = (A + BK)x(k)$$

has the desired eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_n\}$, *i.e.* the characteristic polynomial is

$$\prod_{j=1}^{n} (z - \mu_j) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n = 0 = \Phi(z).$$

Several algorithms are available for this task, *e.g.* Mitter's algorithm, Ackermann's formula (implemented in MATLAB as "*place*").

State space – Design via pole placement

Given a linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

with characteristic polynomial

$$\det(zI - A) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

we want to design a state feedback control law

$$u(k) = -Kx(k)$$

such that the closed-loop system

$$x(k+1) = (A + BK)x(k)$$

has the desired eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_n\}$, *i.e.* the characteristic polynomial is

$$\prod_{j=1}^{n}(z - \mu_j) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n = 0 = \Phi(z).$$

Ackermann's formula

$$K = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}^{-1} \Phi(A)$$

Outline

- Introduction to digital control systems
- Z-transform: definition, properties and theorems
- Sampling and reconstruction
- The pulse transfer function
- Stability and performance
- Control design (discretization, W-plane, root locus and analytical methods)
- State space approach
- Optimal control (dynamic programming and LQR)
- Some advanced topics

Optimal Control - Dynamic programming

Optimal control theory deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved.

A very useful tool to solve (unconstrained) optimal control problems is the **dynamic programming principle**.

Consider the system

$$x(k+1) = f(x(k), u(k)) \qquad k = 0, 1, \dots, N$$

where

- $x(k)$ lives in a finite set X consisting of n elements
- $u(k)$ lives in a finite set U consisting of m elements

and a cost V which we want to minimize

$$\min_{\{u(k)\}} V = \min_{\{u(k)\}} \left\{ C_{\text{terminal}}(x(N), u(N)) + \sum_{k=0}^{N-1} C_{\text{running}}(x(k), u(k)) \right\}$$

Consider the system

$$x(k+1) = f(x(k), u(k)) \qquad k = 0, 1, \ldots, N$$

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Brute Force approach

A naive approach to the solution of the problem if to enumerate all possible trajectories going forward up to time N, calculate the cost for each one, then compare them and select the optimal one.

Since there are m^N possible trajectories and we need N additions to compute the cost, we need around $O(Nm^N)$ algebraic operations to implement this solution. This is highly inefficient!

Consider the system

$$x(k+1) = f(x(k), u(k)) \qquad k = 0, 1, \ldots, N$$

where

- $x(k)$ lives in a finite set X consisting of n elements
- $u(k)$ lives in a finite set U consisting of m elements

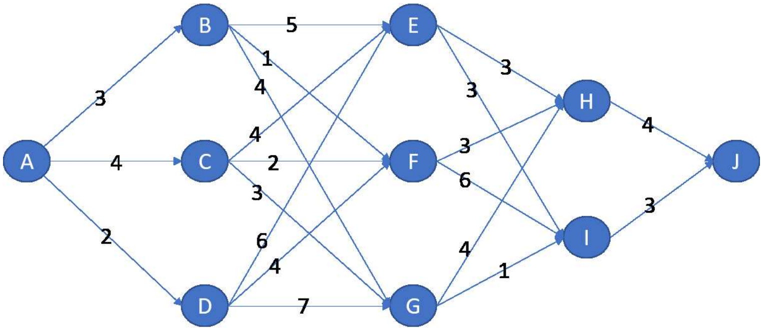
and a cost V which we want to minimize

$$\min_{\{u(k)\}} V = \min_{\{u(k)\}} \left\{ C_{\text{terminal}}(x(N), u(N)) + \sum_{k=0}^{N-1} C_{\text{running}}(x(k), u(k)) \right\}$$

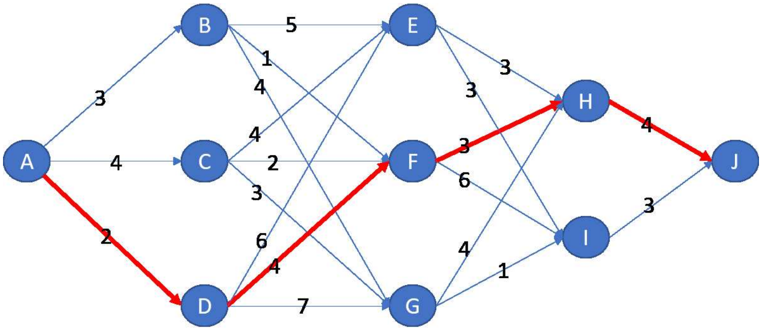
Principle of Optimality (Dynamic Programming)

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (See Bellman, 1957, Chap. III.3.)

We want to go from state x(0)=A to state x(N)=J minimizing the cost (weights on the graph edges).



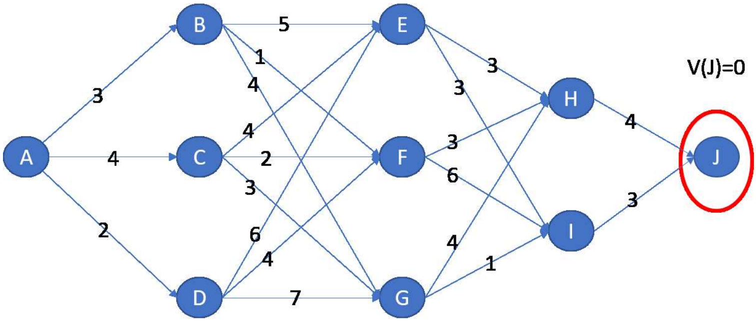
We want to go from state x(0)=A to state x(N)=J minimizing the cost (weights on the graph edges).



Greedy approach: select the least expensive trajectory at each step.

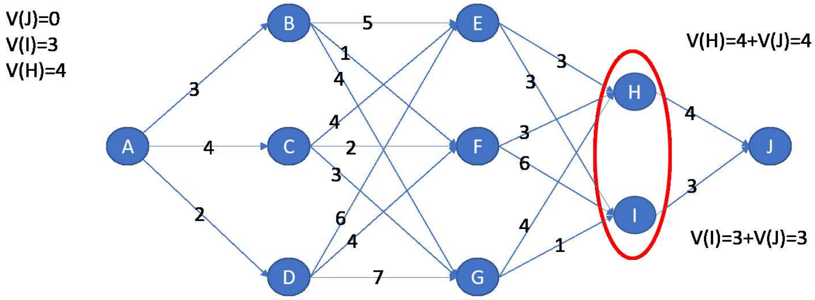
The cost of the Greedy approach is $V = 13$.

We want to go from state x(0)=A to state x(N)=J minimizing the cost (weights on the graph edges).

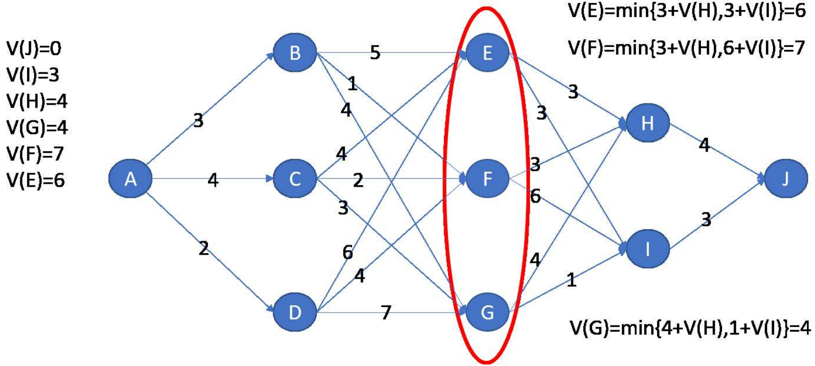


Now we use the dynamic programming principle. We start from the end...

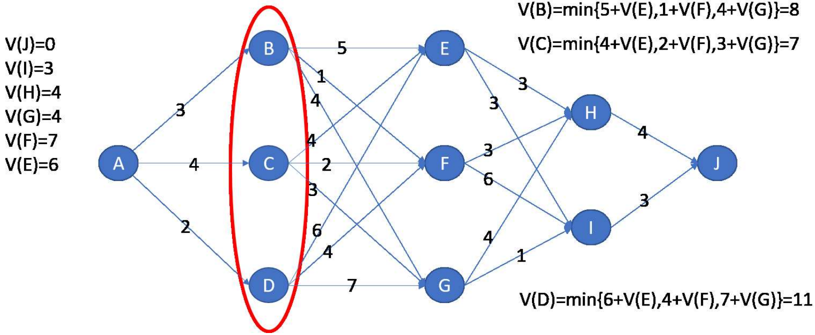
We want to go from state x(0)=A to state x(N)=J minimizing the cost (weights on the graph edges).



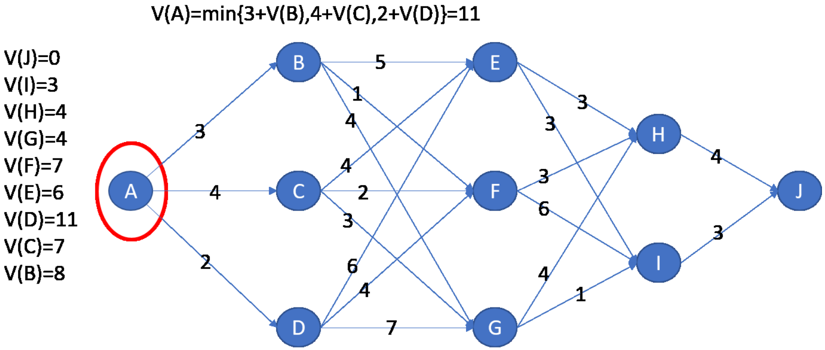
We want to go from state x(0)=A to state x(N)=J minimizing the cost (weights on the graph edges).



We want to go from state x(0)=A to state x(N)=J minimizing the cost (weights on the graph edges).



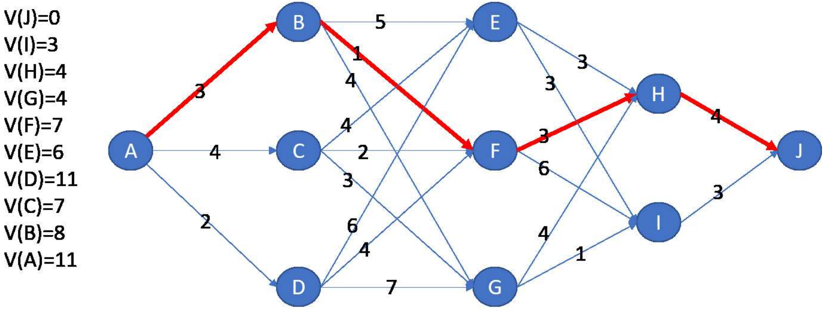
We want to go from state x(0)=A to state x(N)=J minimizing the cost (weights on the graph edges).



So the optimal cost to go from state A to state J is $V = 11$.

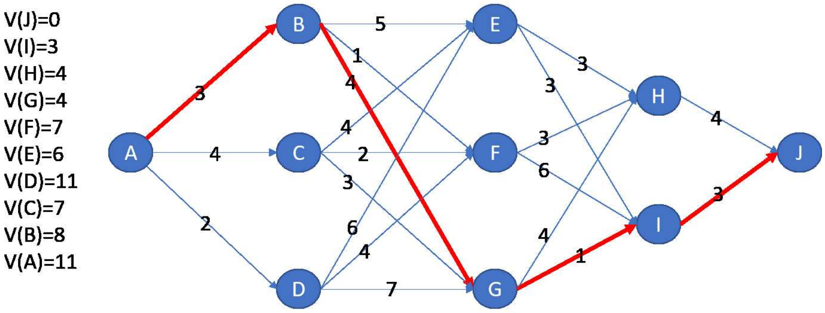
Optimal Control - Dynamic programming

We want to go from state x(0)=A to state x(N)=J minimizing the cost (weights on the graph edges).



One optimal path.

We want to go from state x(0)=A to state x(N)=J minimizing the cost (weights on the graph edges).



The optimal path is not unique.

Computational cost

At each time k, for each state $x(k)$ and each control $u(k)$ we need to add the cost of the corresponding transition to the cost-to-go already computed for the resulting $x(k+1)$.

Thus, the number of required operations is $O(nNm)$. Moreover, this scheme finds the optimal policy for every initial condition $x(0)$. In comparison the brute force approach requires $O(nNm^N)$ operations to achieve the same.

Note that dynamic programming is not useful just from an applicative point of view. It is an important tool also to solve **analytically** optimal control problems, as we do in the next slides.

Optimal Control - LQR

The **linear quadratic regulator** (LQR) problem can be formulated as follows. Given a reachable linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

with $x(0) = x_0$, find the optimal control sequence $\{u(k)\}$ that minimizes the performance index

$$J = x^\top(N)Sx(N) + \sum_{k=0}^{N-1} x^\top(k)Qx(k) + u^\top(k)Ru(k)$$

where

- Q $n \times n$ positive definite or positive semidefinite symmetric matrix
- R $r \times r$ positive definite symmetric matrix
- S $n \times n$ positive definite or positive semidefinite symmetric matrix

Optimal Control - LQR

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$$x(k+1) = Ax(k) + Bu(k)$$

with $x(0) = x_0$, find the optimal control sequence $\{u(k)\}$ that minimizes the performance index

$$J = x^\top(N)Sx(N) + \sum_{k=0}^{N-1} x^\top(k)Qx(k) + u^\top(k)Ru(k)$$

Solution: The optimal control is

$$u(k) = -(R + B^\top P(k+1)B)^{-1}B^\top P(k+1)Ax(k)$$

where the matrix $P(k)$ is the solution of the discrete-time **backward** Riccati equation

$$P(k) = A^\top P(k+1)A + Q - A^\top P(k+1)B(B^\top P(k+1)B + R)^{-1}B^\top P(k+1)A$$

with the initial condition $P(N) = S$.

The optimal cost is $J_{\min} = x_0^\top P(0)x_0$.

Proof via Dynamic Programming

Define the cost-to-go at the time k as

$$V(k) = \min_{u(k),\dots,u(N-1)} \left\{ x^\top(N) S x(N) + \sum_{j=k}^{N-1} x^\top(j) Q x(j) + u^\top(j) R u(j) \right\}$$

Proof via Dynamic Programming

Define the cost-to-go at the time k as

$$V(k) = \min_{u(k),\ldots,u(N-1)} \left\{ x^\top(N)Sx(N) + \sum_{j=k}^{N-1} x^\top(j)Qx(j) + u^\top(j)Ru(j) \right\}$$

Then obviously

$$V(N) = x^\top(N)Sx(N)$$

and we define $P(N) = S$.

Proof via Dynamic Programming

Define the cost-to-go at the time k as

$$V(k) = \min_{u(k),\dots,u(N-1)} \left\{ x^\top(N)Sx(N) + \sum_{j=k}^{N-1} x^\top(j)Qx(j) + u^\top(j)Ru(j) \right\}$$

Then obviously

$$V(N) = x^\top(N)Sx(N)$$

and we define $P(N) = S$.

Now we compute $V(N-1)$ which is

$$V(N-1) \;=\; \min_{u(N-1)} \left\{ V(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\}$$

Proof via Dynamic Programming

Define the cost-to-go at the time k as

$$V(k) = \min_{u(k),\dots,u(N-1)} \left\{ x^\top(N)Sx(N) + \sum_{j=k}^{N-1} x^\top(j)Qx(j) + u^\top(j)Ru(j) \right\}$$

Then obviously

$$V(N) = x^\top(N)Sx(N)$$

and we define $P(N) = S$.

Now we compute $V(N-1)$ which is

$$V(N-1) = \min_{u(N-1)} \left\{ V(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\}$$

Proof via Dynamic Programming

Now we compute $V(N-1)$ which is

$$\begin{aligned} V(N-1) &= \min_{u(N-1)} \left\{ V(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \\ &= \min_{u(N-1)} \left\{ x^\top(N)P(N)x(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \\ &= \min_{u(N-1)} \left\{ (Ax(N-1)+Bu(N-1))^\top P(N)(Ax(N-1)+Bu(N-1)) \right. \\ &\quad \left. + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \end{aligned}$$

Proof via Dynamic Programming

Now we compute $V(N-1)$ which is

$$
\begin{aligned}
V(N-1) &= \min_{u(N-1)} \left\{ V(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \\
&= \min_{u(N-1)} \left\{ x^\top(N)P(N)x(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \\
&= \min_{u(N-1)} \left\{ (Ax(N-1) + Bu(N-1))^\top P(N)(Ax(N-1) + Bu(N-1)) \right. \\
&\quad \left. + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\}
\end{aligned}
$$

Proof via Dynamic Programming

Now we compute $V(N-1)$ which is

$$
\begin{aligned}
V(N-1) &= \min_{u(N-1)} \left\{ V(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \\
&= \min_{u(N-1)} \left\{ x^\top(N)P(N)x(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \\
&= \min_{u(N-1)} \left\{ (Ax(N-1) + Bu(N-1))^\top P(N)(Ax(N-1) + Bu(N-1)) \right. \\
&\quad \left. + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \\
&= \min_{u(N-1)} \left\{ x(N-1)^\top (Q + A^\top P(N)A)x(N-1) + x(N-1)^\top A^\top P(N)Bu(N-1) \right. \\
&\quad \left. + u^\top(N-1)B^\top P(N)Ax(N-1) + u^\top(N-1)(R + B^\top P(N)B)u(N-1) \right\}
\end{aligned}
$$

Proof via Dynamic Programming

Now we compute $V(N-1)$ which is

$$\begin{aligned}
V(N-1) &= \min_{u(N-1)} \left\{ V(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \\
&= \min_{u(N-1)} \left\{ x^\top(N)P(N)x(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \\
&= \min_{u(N-1)} \left\{ (Ax(N-1) + Bu(N-1))^\top P(N)(Ax(N-1) + Bu(N-1)) \right. \\
&\quad \left. + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1) \right\} \\
&= \min_{u(N-1)} \left\{ x(N-1)^\top(Q + A^\top P(N)A)x(N-1) + x(N-1)^\top A^\top P(N)Bu(N-1) \right. \\
&\quad \left. + u^\top(N-1)B^\top P(N)Ax(N-1) + u^\top(N-1)(R + B^\top P(N)B)u(N-1) \right\}
\end{aligned}$$

To this last expression we add and subtract

$$x^\top(N-1)L^\top(N-1)(R + B^\top P(N)B)L(N-1)X(N-1)$$

where

$$L(N-1) = (R + B^\top P(N)B)^{-1}B^\top P(N)A$$

Proof via Dynamic Programming

Now we compute $V(N-1)$ which is

$$
\begin{aligned}
V(N-1) &= \min_{u(N-1)} \left\{V(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1)\right\} \\
&= \min_{u(N-1)} \left\{x^\top(N)P(N)x(N) + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1)\right\} \\
&= \min_{u(N-1)} \left\{(Ax(N-1) + Bu(N-1))^\top P(N)(Ax(N-1) + Bu(N-1)) \right. \\
&\quad \left. + x^\top(N-1)Qx(N-1) + u^\top(N-1)Ru(N-1)\right\} \\
&= \min_{u(N-1)} \left\{x(N-1)^\top(Q + A^\top P(N)A)x(N-1) + x(N-1)^\top A^\top P(N)Bu(N-1) \right. \\
&\quad \left. + u^\top(N-1)B^\top P(N)Ax(N-1) + u^\top(N-1)(R + B^\top P(N)B)u(N-1)\right\}
\end{aligned}
$$

To this last expression we add and subtract

$$x^\top(N-1)L^\top(N-1)(R + B^\top P(N)B)L(N-1)X(N-1)$$

where

$$L(N-1) = (R + B^\top P(N)B)^{-1}B^\top P(N)A$$

and we recognized that the terms in red are quadratic in $u(N-1) + L(N-1)x(N-1)$

Proof via Dynamic Programming

Now we compute $V(N-1)$ which is

$$V(N-1) = \min_{u(N-1)} \{ x(N-1)^\top (Q + A^\top P(N) A - L^\top (N-1)(R + B^\top P(N) B) L(N-1)) x(N-1) +$$
$$+ (u(N-1) + L(N-1)x(N-1))^\top (R + B^\top P(N) B))(u(N-1) + L(N-1)x(N-1)) \}$$

Since quadratic expressions are non-negative, the quadratic expression in $u(N-1) + L(N-1)x(N-1)$ can be minimized when this term is made equal to zero. This is achieved by selecting

$$u(N-1) = -L(N-1)x(N-1)$$

With this selection the cost becomes

$$V(N-1) = x(N-1)^\top (Q + A^\top P(N) A - L^\top (N-1)(R + B^\top P(N) B) L(N-1)) x(N-1)$$

Defining

$$P(N-1) = Q + A^\top P(N) A - L^\top (N-1)(R + B^\top P(N) B) L(N-1)$$

We obtain that

$$V(N-1) = x(N-1)^\top P(N-1) x(N-1)$$

We note that this has the same form of $V(N)$, with N replaced by $N-1$. Hence, we can repeat these same steps for each k.

Proof via Dynamic Programming

The optimal control at time k is

$$u(k) = -L(k)x(k)$$

with

$$L(k) = (R + B^\top P(k+1)B)^{-1}B^\top P(k+1)A$$

and

$$P(k) = A^\top P(k+1)A + Q - L^\top(k)(R + B^\top P(k+1)B)L(k)$$

with the initial condition $P(N) = S$.

The corresponding optimal cost-to-go from x(k) to x(N) is $V(k) = x(k)^\top P(k)x(k)$.

Thus, the optimal cost is $J_{\min} = x_0^\top P(0)x_0$.

Optimal Control - Steady-state LQR

If in the previous problem $N \to \infty$, then the performance index becomes

$$J = \sum_{k=0}^{\infty} x^\top(k)Qx(k) + u^\top(k)Ru(k).$$

The term $x^\top(\infty)Sx(\infty)$ does not appear because if we want that the cost J is finite, then we need $\lim_{k\to\infty} x(k) = 0$ and $\lim_{k\to\infty} u(k) = 0$ *i.e.* the system is asymptotically stable.

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Solution: Since the cost converges to a constant, $\lim_{k\to 0} P(k) = P$, where P is constant. The optimal control is

$$u(k) = -(R + B^\top PB)^{-1}B^\top PAx(k)$$

where the matrix P is the solution of the discrete-time **algebraic** Riccati equation

$$P = A^\top PA + Q - A^\top PB(B^\top PB + R)^{-1}B^\top PA.$$

The optimal cost is $J_{\min} = x_0^\top Px_0$.

Finally note that mathematical manipulations allow writing the Riccati equation in different equivalent forms

$$\begin{aligned} P &= A^\top PA + Q - A^\top PB(B^\top PB + R)^{-1}B^\top PA \\ &= Q - A^\top(P^{-1} + BRB^\top)^{-1}A \end{aligned}$$

Conclusions

This (short) course provides the basic tools for the analysis and design of simple computer-controlled systems.

There are several important issues that we have not discussed.

- Design techniques for standard regulators.
- Advanced design methods.
- Implementation issues and the role of quantization, round-off errors, saturations, ...
- Alternative methods for the construction of discrete-time equivalent models, and methods for the construction of approximate discrete-time models.
- Design methods for nonlinear discrete-time control systems.
- Design methods for hybrid control systems.